Monopolistic Supply of Sorting, Inequality and Welfare*

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Abstract

Why is an increase in income inequality often accompanied by an increase in socio-economic segregation? And what are the welfare implications of this co-movement? This paper uses a theoretical model to analyze the relationship between income inequality and socio-economic segregation. It shows that rising inequality can trigger sorting according to income, as a monopolist's profits from offering sorting increase with income inequality. It also examines the relationship between sorting and social welfare and shows that profit-maximizing sorting patterns are not necessarily optimal from a welfare perspective. In fact, for a broad field of income distributions (monopolist) profits increase with inequality, while at the same time total welfare from sorting decreases.

JEL Classification: D31, D85, Z13, Z18

Keywords: Stratification, Assortative Matching, Group Formation, Income Inequality

*I thank Rabah Amir, Matt Levy, Ronny Razin and the participants of the LSE Microeconomic Theory Work in Progress Seminar for their helpful suggestions and comments.

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1 Introduction

In recent years, we have observed a rise in socio-economic segregation in many industrialized countries.[1] People tend to interact increasingly with others who are not too different from themselves in terms of income, education and political beliefs.[2] Moreover, evidence suggests that segregation and income inequality move jointly. Several studies for the US show that both income inequality and segregation have increased in most metropolitan areas over the past 40 years.[3]

The reasons for this co-movement haven't been explored widely so far. While the presence of assortative matching and (positive) sorting has been extensively discussed in the economics and sociology literature, little research has been done so far on the supply side of segregation and the relationship between inequality and the supply of segregation. Who offers individuals the possibility to segregate and how does this depend on the shape of the income distribution?

Given the trend of mounting socio-economic segregation, an important question is also the social desirability of sorting. If people benefit from interacting with wealthy and influential people, poor people who are deprived of these contacts due to social seclusion will suffer. But sorting might not be universally beneficial for the rich either: Especially if inequality is high, it might be the case that they have to pay huge sums to separate themselves off from the rest of society (e.g. via gated communities or private schools). While Becker (1974) shows that assortative matching always maximizes total surplus in society, Levy and Razin (2015) and Hoppe et al. (2009) demonstrate that segregation is not necessarily beneficial for welfare if we count "sorting fees" as deadweight loss (i.e. costly signals which enable the formation of groups) and subtract them from the surplus.

Finally, it is important to note that the interests of a supplier of the sorting technology might be different from society's interests, and that the way sorting is implemented need not be optimal for society. In addition,

[1]See e.g. Forman and Koch (2013) and Bishop (2008) for evidence on the US.

[2]The terms *segregation* and *sorting* are used interchangibly throughout this paper. Segregation hereby always means *socio-economic* segregation and must not be confused with *racial* segregation or similar concepts.

[3]See e.g. Reardon and Bischoff (2011) and Watson (2009).

an increase in inequality is likely to have different effects on the supplier of the sorting technology and on welfare.

The present paper is a first attempt to analyze the relationship between income inequality and the supply of sorting and to examine how well the interests of the supplier of the sorting technology and of society as a whole are aligned, especially in the face of rising inequality. A simple theoretical model is used in which income is distributed unequally in society and people can pay a "fee" to join a group and interact only with members of that group henceforth. The model is used to examine how this fee will be set if a profit-maximizing monopolist offers the sorting technology, and analyze the monopolist's profits and society's total welfare resulting from this split into groups.

Sorting is not generally beneficial for welfare. In fact, members of the poor group never benefit from being separated from the rest of society. However, depending on the shape of the income distribution, sorting can increase welfare of the rich and their utility gains can compensate for the losses of the poor, such that total welfare in society is higher with sorting compared to random matching. An increase in inequality increases monopolist profits from offering people the possibility to segregate, and potentially also welfare from segregation. However, it turns out that there is often a conflict between welfare and monopolist profits, in the sense that different partitions of society would be optimal for profits and welfare - the way in which the monopolist splits up society is in general not efficient (i.e. welfare maximizing). This conflict tends to intensify as inequality increases: monopolist profits increase, while welfare from sorting decreases as income inequality climbs high. The paper concludes by arguing that there is a sense in which this finding holds also if the monopolist can offer segregation into more than just two groups.

Even when sorting is optimal for total welfare due to large benefits from sorting accruing to the rich, governments would most likely be hesitant to promote segregation because of the losses it entails on the lower end of the income distribution (at least without considering additional measures to redistribute some of the benefits from sorting from the rich to the poor). However, the calculations in this paper show that there are cases in which sorting is detrimental even for total welfare, because the benefits on the upper tail of the income distribution cannot compensate for the losses of the poor, or because even the rich would be better off without sorting. The analysis can thus help to identify situations in which any benevolent social planner (or government agency) should have an interest in counteracting sorting, in particular in the face of rising inequality, for instance via social housing or rent control policies to prevent gentrification or via school (choice) reforms to avoid educational (as well as residential) segregation (see e.g. Xu (2019)).

The presented model can either be read "literally", in the sense that sorting happens in the form of residential segregation (in the extreme even gated communities), private education and exclusive clubs. The sorting fee might then be regarded as the club's membership fee, the cost of private education or the price mark-up for living in a certain area or community. On the other hand, the model can be interpreted along the lines of Rayo (2013), who regards the monopolist as a supplier of conspicuous goods to individuals who want to signal a hidden type, such as their wealth. Those conspicuous goods provide no intrinsic value - consumers want to purchase them simply to obtain social status (via signalling that they belong to a certain "club" of high-ranked individuals who can afford to purchase such goods). Examples might be publicised charity contributions whose cause the donor doesn't care about (see also Glazer and Konrad (1996)) or art purchases when the consumer does not value the piece of art per se.

Both the literal and the more abstract reading of the model justify to some extent the assumption that the sorting technology (or signal) is provided by a monopolist, as both situations are usually associated with a large degree of market power. In particular with conspicuous consumption it is often the case that big and well-established luxury brands have an advantage over small competitors, and consumers tend to coordinate around "fashionable" brands with a certain reputation. But also with the "literal" type of segregation, companies offering residential segregation such as gated communities, but also private schools (think for instance Eton) and exclusive clubs (or similar institutions that enable exclusive interactions, like Scientology) are usually able to act as "quasi" - monopolists given their limited number (at least for a given region). However, the supplier of sorting being a monopolist is clearly just a first, simplifying assumption for an analysis that can provide a starting point for a more thorough investigation into the benefits and welfare implications of sorting and how these depend on inequality.

The rest of the paper is organized as follows: Section 2 presents related literature, Section 3 introduces the model of sorting according to income and examines how changes in inequality affect monopolist profits and welfare. Section 4 demonstrates that there can be a conflict between monopolist profits and welfare as inequality increases. This section is accompanied by an extensive Online Appendix presenting detailed calculations for some of the results. Section 5 examines the effect of increasing inequality on monopolist profits and welfare if the monopolist can offer as many cutoffs as she wants and Section 6 concludes.

2 Related Literature

The standard model of sorting and assortative matching is outlined and analyzed in Becker (1974). Levy and Razin (2015) examine total welfare and preferences for redistribution in the presence of costly income sorting without explicitly modelling the supply side of the sorting technology. Rayo (2013) characterizes optimal sorting if a profit-maximizing monopolist without costs chooses the sorting schedule, while Damiano and Li (2007) analyze the case of two or more competing firms. The present paper carries elements of both Levy and Razin (2015) (in the sense that it analyzes the normative aspects of segregation, in particular its effects on welfare) and of Rayo (2013) (because the sorting technology is assumed to be offered by a profit-maximizing monopolist). The paper's main contribution is a thorough examination of how optimal sorting varies with inequality and how this affects the (potential) conflict between welfare and monopolist profit.

The paper is also related to the literature of educational segregation via private schools (see e.g. Fernandez and Rogerson (2003), Epple and Romano (1998) and Levy and Razin (2017)), as well as the literature on costly signalling (see e.g. Hoppe et al. (2009)) and conspicuous consumption (see e.g. Pesendorfer (1995), Bagwell and Bernheim (1996) and Veblen (1899)). Hopkins and Kornienko (2004) analyze status good consumption and show that as society becomes richer, the proportion of income spent on conspicuous consumption increases and equilibrium utility falls at each level of income. Furthermore, an exogenous decrease in inequality can make the poor worse off. Slack and Ulph (2018) introduce reference consumption into an optimal tax model. They show that for low-productivity workers well-being decreases with net wage, which affects optimal redistributive taxation. In addition, reference consumption is distortionary, which gives a distortion-correcting role to taxation.

Furthermore, the paper is connected to the literature on endogenous jurisdiction formation in connection with sorting (see e.g. Tiebout (1956) and Ellickson (1971)). Gravel and Thoron (2007) examine a situation where unequally wealthy households with identical preferences sort into jurisdictions in order to produce a public good financed by proportional taxation and derive necessary and sufficient conditions for resulting wealth stratification.

Finally, the paper is related to the literature on club formation. While in the present model the utility of belonging to a group increases with own income and the income of the other members, Amir et al. (2014) examine a setup in which club membership, as well as carrying an idiosyncratic intrinsic benefit for each member, becomes more valuable as the size of the club increases. They analyse how positive network externalities affect the formation of thematic clubs and can lead to the disappearance of clubs whose intrinsic value only appeals to a minority of the population. They also compare non-cooperative equilibrium club formation to the optimal club structure a utilitarian social planner would set up and show that the welfare maximizing solution leads to the disappearance of the minority club more often than in the non-cooperative game.

3 Inequality, monopolist profit and welfare

Let income y in an economy be distributed according to an income distribution $F(y)$ with support on the interval $Y = [0, \nu]$ (where $\nu < \infty$ unless explicitly mentioned otherwise). Assume furthermore that $F(y)$ is continuous and strictly monotonic on Y, with pdf $f(y)$ such that $F(z) = \int_0^z f(y)dy$ $\forall z \in Y$. Suppose that an agent's utility is increasing not only in her own income but also in the average income of the people that she interacts with, which will henceforth be called her "reference group". Specifically, a person with income y_j gets utility $U_j = y_j E(y|y \in S_i)$, where S_i is individual j's reference group. If society is not economically segregated, everybody's reference group is a representative sample of the whole population, such that $U_j = y_j E(y)$. However, suppose a person with income y_j can pay a fee $b > 0$ to join group S_b and interact henceforth (mainly) with members of that group. Joining group S_b yields utility

$$y_j E[y|y \in S_b] - b,$$

but the agent can also refrain from paying b and get utility

$$y_j E[y|y \in S_0]$$

instead, where S_b is the set of incomes y of people who have paid b and S_0 is the set of incomes y of people who haven't paid b. Then we can define the following:

Definition 1 *A sorting equilibrium is a partition* $[S_0, S_b]$ *of* Y *and a sorting fee* $b > 0$ *such that*

$$yE[y|y \in S_b] - b \leq yE[y|y \in S_0] \quad \forall y \in S_0 \tag{1}$$

$$yE[y|y \in S_b] - b \geq yE[y|y \in S_0] \quad \forall y \in S_b \tag{2}$$

In a sorting equilibrium as defined above, people stay in the group that gives them the highest utility.

Appendix A.1 shows that in any sorting equilibrium, group S_b must have a higher average income than group S_0, and that all sorting equilibria will be *monotone,* meaning that the groups S_0 and S_b are single intervals of Y (where group S_b must lie to the right of group S_0 on the Y scale). Therefore, I will from now on call people in S_b "the rich" and people in S_0 "the poor".

Furthermore, as all equilibria are monotone, the definition of a sorting equilibrium can be rewritten in terms of a cutoff $\hat{y}$, where everybody with income below the cutoff is in the poor group and everybody with income above the cutoff is in the rich group. For simplicity of notation, average income in the rich group, $E[y|y \in S_b]$, will be denoted by $\bar{E}(\hat{y})$ and average income in the poor group, $E[y|y \in S_0]$, by $\underline{E}(\hat{y})$.[4] Appendix A.1 shows the following:

Proposition 1 *Any sorting equilibrium is characterized by a cutoff* $\hat{y} \in Y$ *and a sorting fee* $b > 0$ *such that*

$$\hat{y}\bar{E}(\hat{y}) - \hat{y}\underline{E}(\hat{y}) = b \tag{3}$$

Proof. See Appendix A.1. ■

A person with income $\hat{y}$ just at the border of the two groups S_b and S_0 has to be exactly indifferent between joining either of the two groups in equilibrium. For the remainder of the paper I will choose the convention that people with income $\hat{y}$ (who are indifferent between the two groups) stay in the poor group.

It can immediately be seen from (3) that the sorting fee is uniquely determined by the equilibrium cutoff $\hat{y}$, i.e. for a given equilibrium partition $\{[0, \hat{y}], (\hat{y}, \nu]\}$, the sorting fee b is unique. The reverse statement is not true in general: For a given b, there might be multiple cutoffs $\hat{y}$ that satisfy $\hat{y}(\bar{E}(\hat{y}) - \underline{E}(\hat{y})) = b$ (this could happen if the distribution is such that $\hat{y}(\bar{E}(\hat{y}) - \underline{E}(\hat{y}))$ is not strictly increasing or decreasing for all $\hat{y} \in Y$).[5] For a given sorting fee, there could therefore be several monotone partitions of society that would be sorting equilibria given this fee. When modelling the supply side below, I thus require that whoever offers the sorting technology chooses the cutoff optimally and I implicitly assume that the supplier can then ensure that the agents coordinate on the equilibrium that yields the highest payoff for the supplier (which, in the case of a profit-maximizing firm, would always be the lowest cutoff $\hat{y}$ such that $\hat{y}(\bar{E}(\hat{y}) - \underline{E}(\hat{y})) = b$, because it yields the largest mass of customers).

3.1 Monopolist profit

The model outlined above shows how the sorting fee has to be set in order to generate a certain partition of society. But who determines how the

[4]Note that $\bar{E}(\hat{y}) = \frac{\int_{\hat{y}}^{\nu} yf(y)dy}{1-F(\hat{y})}$ and $\underline{E}(\hat{y}) = \frac{\int_0^{\hat{y}} yf(y)dy}{F(\hat{y})}$.

[5]It can be shown that a sufficient condition for $\hat{y}(\bar{E}(\hat{y}) - \underline{E}(\hat{y}))$ to be monotone is that the income distribution is *new worse than used in expectations (NWUE).* For a definition of the NWUE property see Section 3.2.

groups in society look like? Who offers the sorting technology and chooses the cutoff? As mentioned in the introduction, both the "literal" reading of the model concerning residential segregation, private education and clubs, and the "signalling" interpretation along the lines of Rayo (2013) point towards some degree of market power on the supply side. For the remainder of this paper I will thus assume that the sorting technology is offered by a profit-maximizing monopolist and examine the implications of an increase in inequality for the monopolist's profits and for total welfare.

The next sections will focus on the model of sorting with two groups as described above. The monopolist can therefore only decide between offering one cutoff or staying inactive, but she cannot offer more than one cutoff. This could be modelled explicitly by assuming that the costs of offering more than one cutoff are prohibitively high. The last section of this paper discusses what happens if the monopolist's costs are negligible and she can therefore offer as many cutoffs as she wants.

If the monopolist faces fixed costs $c > 0$ of operating, her profits from offering sorting are

$$\Pi(\hat{y}^*) = R(\hat{y}^*) - c,$$

where $R(\hat{y}^*)$ is the revenue from offering sorting at cutoff $\hat{y}^*$ and $\hat{y}^*$ is chosen optimally, such that

$$\hat{y}^* = \arg\max_{\hat{y}} R(\hat{y}).$$

Revenue at cutoff $\hat{y}$ is given by

$$R(\hat{y}) = \hat{y}(\bar{E}(\hat{y}) - \underline{E}(\hat{y}))(1 - F(\hat{y})) = \hat{y}(\bar{y} - \underline{E}(\hat{y})),$$

where $\bar{y}$ is average income. It is straightforward to see that any solution to the revenue maximization problem must be interior, because $R(0) = R(\nu) = 0$ whereas $R(\hat{y})$ is strictly positive for any interior $\hat{y}$. The monopolist's optimal cutoff is thus implicitly defined via the first order condition,[6]

$$R'(\hat{y}^*) = \bar{y} - \underline{E}(\hat{y}^*) - \hat{y}^* \frac{\partial \underline{E}(\hat{y}^*)}{\partial \hat{y}} = 0. \tag{4}$$

Suppose that the income distribution and the fixed costs c are such that $\Pi(\hat{y}^*) > 0$ and hence it is profitable for the monopolist to offer the sorting technology. What happens to her profits as inequality increases? The following proposition states that the monopolist's profits always rise if inequality increases in the form of a particular type of mean-preserving spread of the income distribution. I shall say that a mean-preserving spread is *monotone* if $\bar{E}(\hat{y})$ increases and $\underline{E}(\hat{y})$ decreases for any interior cutoff $\hat{y}$ (while of course, as implied by the definition of a mean-preserving spread, average income $\bar{y}$ doesn't change). Note that any mean-preserving spread of the income distribution necessarily implies an increase in the Gini coefficient (see Dalton (1920) and Cowell (2000)), and thus in inequality.

Proposition 2 *An* ***increase in inequality*** *in the form of a monotone mean-preserving spread of the income distribution* ***increases*** *the monopolist's profits from offering sorting.*

Proof. If inequality increases in the form of a monotone mean-preserving spread of the income distribution, the difference $\bar{y} - \underline{E}(\hat{y})$ will increase. This implies a rise in $\hat{y}^*(\bar{y} - \underline{E}(\hat{y}^*))$, keeping $\hat{y}^*$ constant at the optimal choice for the initial income distribution. It is very likely that the optimal cutoff will also change for the monopolist, but even with keeping the old cutoff, her revenues increase, and they will do even more so if the monopolist also chooses the cutoff optimally. ■

It is easy to see from the above proof that, in order for the monopolist's profits to increase, the mean-preserving spread does not have to be such that $\bar{E}(\hat{y})$ increases and $\underline{E}(\hat{y})$ decreases for *any* cutoff - it suffices if this holds for the initially optimal cutoff. Proposition 2 therefore states sufficient conditions for an increase in the monopolist's profits.

[6] As $\underline{E}(\hat{y}) = \frac{\int_0^{\hat{y}} y f(y) dy}{F(\hat{y})}$, we get that $\frac{\partial \underline{E}(\hat{y}}{\partial \hat{y}} = \frac{f(\hat{y})(\hat{y} - \underline{E}(\hat{y}))}{F(\hat{y})}$ and thus (4) can be written as

$$\bar{y} - \underline{E}(\hat{y}^*) = \hat{y}^* \frac{f(\hat{y}^*)(\hat{y}^* - \underline{E}(\hat{y}^*))}{F(\hat{y}^*)}.$$

If the income distribution and the fixed cost are initially such that $\Pi(\hat{y}^*) < 0$, an increase in inequality can have an effect on the monopolist's decision of whether or not to offer sorting at some $\hat{y}$, where she compares the profits from offering the sorting technology to 0 (the profits she would make if she stays inactive). An increase in inequality of the form described above, if it is large enough, will make the monopolist's profits positive, which in turn leads the monopolist to become active. As a result, society can become economically segregated due to an increase in inequality in the form of a mean-preserving spread of the income distribution.

Corollary 1 *If society is not segregated initially, a* ***sufficiently high increase in inequality*** *in the form of a monotone mean-preserving spread will make it profitable for a monopolist to offer sorting and will thus* ***trigger segregation****.*

A mean-preserving spread is not the only type of increase in inequality that increases the monopolist's profits from offering sorting. In fact, from examining the expression for the monopolist's profits, $\hat{y}^*(\bar{y} - \underline{E}(\hat{y}^*)) - c$, it is straightforward to see that any increase in inequality that increases $\bar{y} - \underline{E}(\hat{y}^*)$ for the initially optimal cutoff $\hat{y}^*$ will raise the monopolist's profits.

3.2 Welfare

The above section shows that an increase in inequality in the form of a mean-preserving spread increases a monopolist's profit from offering sorting. But what happens to welfare in society? Total welfare under no sorting (and thus random matching) is simply

$$W^P = \int_0^{\nu} y\bar{y}f(y)dy = \bar{y}^2.$$

Total welfare from sorting at cutoff $\hat{y}$ can be written as[7]

$$W(\hat{y}) = \underline{E}(\hat{y})\int_0^{\hat{y}} yf(y)dy + \bar{E}(\hat{y})\int_{\hat{y}}^{\nu} yf(y)dy - (1-F(\hat{y}))\hat{y}(\bar{E}(\hat{y}) - \underline{E}(\hat{y})) \quad (5)$$

$$= F(\hat{y})(\underline{E}(\hat{y}))^2 + (1 - F(\hat{y}))(\bar{E}(\hat{y}))^2 - \hat{y}(\bar{y} - \underline{E}(\hat{y}))$$

It is immediate to see that sorting is never beneficial for the poor group - they would always obtain higher utility on average if they could interact randomly with everybody in society. However, welfare can be higher for people in the rich group (and in particular for the richest among those) compared to random matching, despite the sorting fee they have to pay in order to separate themselves off from the rest of society. Whether total welfare, which is just the sum of utilities in the poor and the rich group, is higher under sorting compared to random matching depends on the shape of the income distribution (and, in general, on the cutoff).

Levy and Razin (2015) characterize distributions for which sorting is always more efficient (in a utilitarian sense, i.e. yielding higher total welfare) than no sorting, irrespective of the cutoff. They show that the difference between welfare of sorting at cutoff $\hat{y}$ and welfare of no sorting can be written as

$$W(\hat{y}) - W^P = (\bar{y} - \underline{E}(\hat{y}))(\bar{E}(\hat{y}) - \bar{y} - \hat{y}) \quad (6)$$

and thus sorting yields higher total welfare than no sorting for any $\hat{y}$ iff the income distribution is such that

$$\bar{E}(\hat{y}) - \bar{y} > \hat{y} \quad \forall \hat{y}. \quad (7)$$

This condition is what in reliability theory has been termed the *new worse than used in expectations (NWUE)* property. A distribution F is NWUE if condition (7) is satisfied, and *new better than used in expectations (NBUE)*

[7]As in Levy and Razin (2015), welfare from sorting at a particular cutoff takes into account the sorting fee paid as a deadweight loss to society, or benefiting only a negligible proportion of society (which is the case if the supplier is a monopolist). If the sorting fee would not be considered, perfect sorting would always yield maximal total welfare, because the utility from a match is supermodular (see Becker (1974)).

if the opposite holds, i.e.

$$\bar{E}(\hat{y}) - \bar{y} < \hat{y} \quad \forall \hat{y}.$$

It is immediate to conclude the following:

1. If F is NWUE, sorting at any cutoff yields higher total welfare than no sorting.
2. If F is NBUE, no sorting yields higher total welfare than sorting at any $\hat{y}$.
3. If F is not NBUE, then there will always exist some cutoff $\hat{y}$ at which sorting yields higher total welfare than no sorting.

To see the intuition behind 1. and 2., note that the positive assortativity benefits from sorting outweigh the welfare costs (in form of the sorting fee that needs to be paid) if the distribution is sufficiently unequal (such that random matching would lead to much lower utilities for rich individuals). NWUE distributions exhibit a greater degree of variability than NBUE distributions.[8] Hence, the utility benefits of sorting outweigh the costs for NWUE distributions, while the opposite holds in the NBUE case.

In light of these considerations, it is not surprising that a monotone mean-preserving spread increases total welfare at certain cutoffs $\hat{y}$.

Proposition 3 *An **increase in inequality** in the form of a monotone mean-preserving spread of the income distribution **increases total welfare** from sorting at those cutoffs where $\bar{E}(\hat{y}) - \bar{y} > \hat{y}$.*

Proof. If $\bar{E}(\hat{y}) - \bar{y} > \hat{y}$ then (6) tells us that the difference between welfare of sorting at $\hat{y}$ and welfare of no sorting increases due to this mean-preserving spread (both $\bar{y} - \underline{E}(\hat{y})$ and $\bar{E}(\hat{y}) - \bar{y} - \hat{y}$ increase). As welfare of no sorting is $\bar{y}^2$ and thus doesn't change due to a mean-preserving spread, this implies that welfare of sorting at $\hat{y}$ must increase. ■

From Proposition 3 we can immediately deduce the following

[8] All NWUE distributions have a coefficient of variation $CV = \frac{\sqrt{Var(y)}}{\bar{y}}$ larger than (or equal) 1, whereas the NBUE property implies that $CV \leq 1$ (see Hall and Wellner (1981)).

Corollary 2 *If F is NWUE, an* ***increase in inequality*** *in the form of a monotone mean-preserving spread of the income distribution* ***increases total welfare*** *from sorting at any given cutoff.*

Note that from Proposition 3 no general predictions can be made for welfare at those cutoffs where $\bar{E}(\hat{y}) - \bar{y} < \hat{y}$: On the one hand, $\bar{y} - \underline{E}(\hat{y})$ increases, but on the other hand $\bar{E}(\hat{y}) - \bar{y} - \hat{y}$ is negative (even though the mean-preserving spread will decrease this term in absolute value). The total effect of the mean-preserving spread on (6) is thus ambiguous and will depend on the shape of the analyzed income distribution.

If F is NBUE and hence there *is* no cutoff such that $\bar{E}(\hat{y}) - \bar{y} > \hat{y}$, a mean-preserving spread can make sorting efficient (i.e. better for welfare than no sorting) for some cutoffs.

Proposition 4 *If F is initially NBUE, a sufficiently large monotone mean-preserving spread of the income distribution will make sorting efficient at some cutoff $\hat{y}$.*

Proof. The mean-preserving spread will increase $\bar{E}(\hat{y}) - \bar{y}$ for all $\hat{y}$, which will eventually make $\bar{E}(\hat{y}) - \bar{y} - \hat{y}$ positive for some $\hat{y}$. ■

An increase in inequality will therefore increase total welfare from sorting at those cutoffs for which $\bar{E}(\hat{y}) - \bar{y} - \hat{y} > 0$ and can make sorting at *some* cutoff efficient (from a utilitarian point of view) if F is initially NBUE. Importantly, though, it is not necessarily the case that sorting *at the cutoff that the monopolist chooses* after the increase in inequality yields higher welfare than before As described above, a mean-preserving spread of the income distribution increases welfare of sorting at those cutoffs for which $\bar{E}(\hat{y}) - \bar{y} - \hat{y} > 0$, but what happens to total welfare from sorting at the other cutoffs depends on the shape of the income distribution. Furthermore, even if the monopolist's optimal cutoff is initially such that $\bar{E}(\hat{y}) - \bar{y} - \hat{y} > 0$, the change in the shape of the income distribution can imply that the monopolist chooses a different cutoff after the mean-preserving spread, at which total welfare is lower than before.

Finally, is important to emphasize that, as mentioned above, even if the income distribution is such that sorting yields higher total welfare than random matching, the poor group never benefits from sorting. Sorting can be efficient (in a utilitarian sense) for some income distributions (in particular for those which exhibit the NWUE property) due to large utility gains for people on the upper end of the income distribution, who profit from not having to interact with the poor part of society. However, poor people suffer from sorting because they lose the possiblity to interact with high-income individuals. Hence, even though from a utilitarian perspective a social planner should be in favour of sorting whenever it maximizes social welfare, this is probably not what any real-life government agency would promote (at least not without considering additional measures aimed at redistributing some of the accruing benefits from sorting from the rich to the poor). But the calculations in this section and the following can identify cases in which sorting (at any cutoff, or at the monopolist's optimally chosen one) is detrimental even for total welfare, because the utility gains accruing to the rich cannot compensate for the losses of the poor (or because even the rich would be better off without sorting). The analysis can thus help pinpoint situations in which sorting is a particulary bad idea from a welfare perspective and any benevolent social planner (or government agency) should have an interest in counteracting it, in particular as inequality increases (which, as Proposition 2 demonstrates, ups the attractiveness of offering sorting from a monopolist's point of view).

The relationship between the monopolist's profit and welfare and how this relationship changes as inequality increases will be the focus of the next section.

4 Increasing inequality and the conflict between monopolist profit and welfare

The analysis so far shows that an increase in inequality in the form of a monotone mean-preserving spread always increases the monopolist's profit and can also raise total welfare from sorting for some cutoffs $\hat{y}$. However, the cutoffs at which monopolist profits increase are not necessarily the same as the ones where welfare increases. Indeed, a monopolist's and a utilitarian social planner's interests are in general not aligned. As demonstrated below, total welfare from sorting at the monopolist's optimal cutoff often declines with inequality.

Section 1 in the Online Appendix analyzes a stylized income distribution to illustrate the potential conflict between monopolist profits and welfare due to increasing inequality. I decided to call this distribution the *symmetric atoms distribution*.[9] In the Online Appendix I show that (unless inequality is very small) total welfare at the monopolist's optimal cutoff is decreasing in inequality, and that both welfare of the richest in society and average welfare in the poor group decline with rising inequality. The following section examines, which of these findings apply to a more general class of distributions.

First, five stylized types of income distributions are analyzed. These distributions have the same average income, but differ in their implied degree of inequality (measured as $\bar{E}(\hat{y}) - \underline{E}(\hat{y})$ for any cutoff). I analyze what these different degrees of inequality imply for monopolist profits and resulting total welfare. The income distributions range from full equality (where everybody in society has the same income) to a distribution that could be classified as "high inequality" (where half of the population have nothing, and half have the maximum possible income).

Note that all of these distributions are NBUE (see Appendix A.2.2) and thus no sorting yields higher total welfare than sorting at any cutoff (except for the case of total equality, where sorting and no sorting yield the same welfare). Furthermore, as average income is the same for all the distributions in this analysis, total welfare without sorting doesn't vary with inequality and is $\frac{\nu^2}{4}$ in all cases.[10]

- **Egalitarian distribution**

  If the income distribution is completely egalitarian, i.e. everybody has income $\frac{\nu}{2}$, then the monopolist's profits from sorting will be 0 (because offering sorting will not be profitable with fixed costs or yield a profit of 0 without fixed costs). Total welfare in this case is $\frac{\nu^2}{4}$.

[9]This distribution, and also some of the distributions analyzed later in this paper, don't satisfy all the conditions required in the initial setup of the model, i.e. $F(.)$ is in general not continuous and strictly monotonic. However, this is not a problem for most of the below calculations.

[10]All the background calculations for the results below can be found in Section 2 of the Online Appendix.

- **Triangle distribution**

  If income is distributed in a triangular (isosceles) shape on $[0, \nu]$ such that the density is

$$
\begin{aligned}
f(y) &= \frac{4}{\nu^2}y \qquad \text{if } y \in \left[0, \frac{\nu}{2}\right] \\
f(y) &= \frac{4}{\nu} - \frac{4}{\nu^2}y \qquad \text{if } y \in \left[\frac{\nu}{2}, \nu\right],
\end{aligned}
$$

  the profit-maximizing cutoff for the monopolist is $\hat{y}^* = \frac{3\nu}{8}$ and the resulting profits are $\frac{3\nu^2}{32}$. Total welfare from sorting at this cutoff amounts to $\frac{3059}{529}\frac{\nu^2}{32} < \frac{\nu^2}{4}$.

- **Uniform distribution**

  If income is uniformly distributed on $[0, \nu]$, the monopolist's profit maximizing cutoff is $\hat{y}^* = \frac{\nu}{2}$ and the resulting profit is $\frac{\nu^2}{8}$. Total welfare from sorting at this cutoff is $\frac{3\nu^2}{16} < \frac{\nu^2}{4}$.

- **Reverse triangle distribution**

  If income is distributed in a reverse-triangular (isosceles) shape on $[0, \nu]$ such that the density is

$$
\begin{aligned}
f(y) &= \frac{2}{\nu} - \frac{4}{\nu^2}y \qquad \text{if } y \in \left[0, \frac{\nu}{2}\right] \\
f(y) &= -\frac{2}{\nu} + \frac{4}{\nu^2}y \qquad \text{if } y \in \left[\frac{\nu}{2}, \nu\right]
\end{aligned}
$$

  the monopolist's optimal cutoff is $\hat{y}^* = 0.64\nu$, which yields a profit of $0.1935\nu^2$. Total welfare from sorting at this cutoff is $0.163\nu^2 < \frac{\nu^2}{4}$.

- **Binary inegalitarian distribution**

  If half of the population has 0 income and half of them earn ν, the optimal cutoff for the monopolist is $\hat{y} = \nu$ with corresponding sorting fee $\frac{\nu}{2}$. Note that due to the jump in F at ν (F is not continuous here!) the sorting fee is not uniquely determined, any $b \in (0, \frac{\nu}{2}]$ would work, and the monopolist will choose the highest in this interval to maximize her profits (and therefore the profits will be $\frac{\nu^2}{2}$). Resulting total welfare in this case would be 0. Welfare would be maximized with the same partition, i.e. a poor group with zero income and a rich group with income ν, but with the lowest of feasible sorting fees, i.e. b being just ϵ over 0. Resulting welfare would be $\frac{\nu}{2} - \frac{\epsilon}{2}$. If the mass at both endpoints is not equal, this last result holds as well, because it is always efficient to separate rich and poor if the sorting fee is negligible, due to the supermodularity of utility from sorting (see Becker (1974)). The monopolist's profit in that latter case is increasing in the mass of rich people relative to poor people.

From this simple analysis the following can be concluded for these five distributions:

1. As **inequality increases** (in terms of discrete jumps from one distribution to another), **the monopolist's profits increase**.

2. As inequality increases, the monopolist's optimal cutoff increases.

3. Total welfare is independent of inequality in the absence of sorting, it depends only on average income. For all the above analyzed distributions, no sorting is more efficient than sorting at any cutoff $\hat{y}$.

4. If the monopolist chooses the cutoff to maximize her profits, then total welfare is highest in the case of the egalitarian distribution (because the sorting fee is 0 in that case and the situation is equal to no sorting, which maximizes welfare for all the distributions discussed above). The next highest welfare would be achieved in the uniform case, followed by the triangular and then the reverse triangular case, and the binary inegalitarian distribution would be worst for welfare (given the sorting fee that the monopolist would charge). Hence - if we exclude the case of the egalitarian distribution and start from a triangular distribution - **total welfare from sorting** at the monopolist's optimal cutoff **initially increases with inequality**, but **as inequality climbs too high** the monopolist can claim a huge part of the gross benefits from sorting for herself and **total welfare decreases**.

For the symmetric atoms distribution, Section 1 in the Online Appendix shows that welfare from sorting at the monopolist's optimal cutoff is decreasing in inequality. Here, we see that if we don't only look at mean-preserving spreads of the uniform distribution, but start from a situation where inequality is smaller than for a uniform distribution, the picture is different: Welfare increases with inequality for small rates of inequality, and decreases thereafter. Section 2.4 in the Online Appendix shows that this is true not only for the above discrete jumps in inequality but also for continuous changes in inequality for these types of distributions. In particular, the Online Appendix analyzes a distribution that is, for low levels of inequality, shaped like a house, and then as inequality increases becomes uniform and in the end looks like a reverse house (or trough). The two extreme cases are thus the triangle distribution (low inequality) and the reverse triangle distribution (high inequality) from above. The analysis finds the same results for this continuous version of the stylized distributions above: welfare of sorting at the monopolist's optimal cutoff y^* increases in inequality for low rates of inequality, and decreases for high rates. In a sense, there is thus less of a conflict between profit maximization and welfare for low rates of inequality than for high rates. However, note that all these distributions, ranging from the triangle to the reverse triangle one and all degrees of inequality in between, are NBUE and hence no sorting yields higher welfare than sorting at any cutoff (see Appendix A.2.2). Even though, for low rates of inequality, an increase in inequality increases total welfare from sorting at the monopolist's optimal cutoff, random matching would be superior from a utilitarian perspective in all those cases.

5 Multiple groups

The previous sections examine how increasing inequality affects welfare and profits if the monopolist can choose one cutoff and thus offer segregation into two groups. The above analysis shows that the interests of a profit-maximizing monopolist and a utilitarian social planner are generally not aligned, and that the conflict between those interests increases with inequality. In the following section these results are compared to a situation where the monopolist doesn't face any costs of offering segregation and can therefore offer infinitely many groups (i.e. perfect sorting) if she wants. It turns out that the findings from the previous sections hold in some sense also for this more general setting: There is a way in which an increase in inequality increases the conflict between monopolist's profits and welfare (and lets the monopolist extract more surplus, if she can set the menu of sorting fees).

Before looking at the monopolist's optimization problem, let me first examine what is best for total welfare if multiple groups are possible.[11] In a first step we can compare welfare from no sorting to welfare arising from perfect sorting. Hoppe et al. (2009) show that if the income distribution is such that the coefficient of variation, which is given by

$$CV(y) = \frac{\sqrt{Var(y)}}{\bar{y}},$$

is larger than 1 then perfect sorting yields higher welfare than no sorting, and if $CV(y) \leq 1$, the opposite holds:

Proposition 5 *(Hoppe et al. (2009)) Perfect sorting is more (less) efficient than no sorting iff* $CV(y) \geq (\leq) 1$.

Proof. The proof from Hoppe et al. (2009) is reproduced in Section 3.1 of the Online Appendix. ■

The coefficient of variation can be regarded as a measure of inequality - it is high if the difference between the standard deviation and the average is high, and it increases due to a mean-preserving spread of the income distribution. Hence, another way to interpret Proposition 5 is in terms of inequality: For low rates of inequality, no sorting is more efficient than perfect sorting, whereas if inequality is high, perfect sorting yields higher welfare than random matching. The intuition behind this result is that the decision between pooling the entire population and perfect sorting involves a trade-off: Rich people achieve lower utility if they have to interact with the whole population than if they can separate themselves off, while

[11] Note again that examining welfare in society as just the sum of individual utilities (i.e. total welfare) does not shed light onto who in society benefits or loses from sorting. With two groups (like in the previous sections), it is clear that the poor group is always worse off with sorting compared to random matching, whereas it depends on the shape of the income distribution whether the rich benefit from sorting. With multiple groups, matters are not so simple. Depending on how the groups are chosen (and, again, on the shape of the income distribution), even people at the lower end of the income distribution could be better off with sorting compared to random matching, if the sorting schedule is such that it prevents them from interacting with poorer people.

poor people attain higher utility if they can interact with the rich. For high rates of inequality, the former effect dominates the latter due to the supermodularity of the utility function.

It can be shown that the NBUE property, introduced in the previous section, is intricately connected to another property for distributions called IFR. A distribution exhibits an *increasing failure rate (IFR)* iff $\frac{f(y)}{1-F(y)}$ is increasing for all y. Two things can be shown for distributions with IFR:

Proposition 6

- *If $F(y)$ is IFR $\implies$ $F(y)$ is NBUE (Marschall and Proschan (1972))*
- *IF $F(y)$ is NBUE $\implies$ CV$(y) \leq 1$ (Hall and Werner (1981))*

Corollary 3 *All IFR distributions must have a coefficient of variation smaller than* 1.

Proof. The proof of the first part of Proposition 6 is a well-know fact from reliability theory, proved for instance in Marshall and Proschan (1972). Hall and Wellner (1981) show that if $F(y)$ is NBUE then the coefficient of variation is smaller than unity. Together, these two facts imply that all IFR distributions must have a coefficient of variation smaller than 1. ∎

Appendix A.2.2 shows that the triangle distribution, the uniform distribution and the reverse triangle distribution discussed in the previous section and the distribution discussed in the Online Appendix (which encompasses all the others) are NBUE. As the coefficient of variation is smaller than 1 for all distributions which are NBUE, no sorting yields higher welfare than perfect sorting for these distributions. The symmetric atoms distribution is not NBUE - indeed Appendix A.2.1 shows that for small $\hat{y}$ sorting yields higher welfare than no sorting. However, the symmetric atoms distribution has $CV(y) \leq 1$ and therefore perfect sorting always yields lower welfare than no sorting (see Appendix A.2.1).

For NBUE distributions, Levy and Razin (2015) also prove an additional feature:

Proposition 7 *(Levy and Razin (2015)) No sorting yields higher welfare than any finite sorting (i.e. any finite incentive compatible partition of society) iff $F(y)$ is NBUE.*

Proof. The main intuition behind the proof in Levy and Razin (2015) is reproduced in Section 3.2 of the Online Appendix. ■

For the class of NBUE distributions, we can hence conclude the following:

Corollary 4 *If the income distribution is NBUE, no sorting yields higher welfare than perfect sorting or any type of finite sorting.*

Proof. This follows immediately from Propositions 5, 6 and 7. ■

Note that, as NBUE is a weaker property than IFR, Corollary 4 also holds for the class of IFR distributions.

After characterizing the class of distributions for which no sorting yields higher welfare than perfect sorting and any finite sorting, we can now analyze the monopolist's optimization problem: What is the monopolist's optimal sorting schedule if she doesn't face any costs of offering the technology? Rayo (2013) characterizes the optimal placement of regions of pooling and perfect sorting, depending on the shape of the income distribution.[12] In the following I want to examine the implications of changing inequality on the monopolist's optimal sorting schedule and total welfare.

Rayo shows that if (and only if) the function $h(y) = y - \frac{1-F(y)}{f(y)}$ is nondecreasing everywhere, perfect sorting is the profit-maximizing sorting schedule. If there are regions of y for which $h(y)$ is decreasing, perfect sorting is not optimal for the monopolist and she will want to introduce intervals of y for which she pools everybody into one joint group.[13]

It is immediate to see that $h(y)$ is always decreasing if the distribution has an increasing failure rate (IFR). Hence, if a distribution exhibits IFR, perfect sorting is optimal for the monopolist. We can therefore conclude the following:

Corollary 5 *If the income distribution exhibits IFR, a monopolist and a utilitarian social planner have conflicting interests: No sorting is more efficient than perfect sorting or any type of finite sorting, but the monopolist wants to implement perfect sorting to maximize her profits.*

[12]Section 3.3 of the Online Appendix reproduces the main intuition behind his findings.

[13]Note however that there are never two pooling intervals next to each other (i.e. pooling intervals are always maximal) and that pooling is never optimal at the top end of the distribution.

What happens within the class of distributions for which perfect sorting is optimal for the monopolist (note that this class contains the family of IFR distributions, which are characterized by low inequality in terms of the coefficient of variation, because $IFR \Rightarrow CV \leq 1$) if inequality increases in the sense of a mean-preserving spread of the income distribution? We know that total welfare and monopolist profit are both

$$\int \frac{y^2}{2} f(y)dy = \frac{E(y^2)}{2}$$

under perfect sorting, i.e. both the monopolist and the citizens get half of the total surplus from perfect sorting (see Appendix A.2.1). Hence, whenever a change to the distribution happens such that perfect sorting is still optimal for the monopolist afterwards, welfare and profits are affected in the same way, i.e. a utilitarian social planner's and a monopolist's interests are aligned. For instance, look at the effects of a mean-preserving spread: The variance increases but average income doesn't change. Because of

$$Var(y) = E(y^2) - \bar{y}^2$$

this implies that $E(y^2)$ must increase due to a mean-preserving spread, which means that a mean-preserving spread increases both total welfare and the monopolist's profits in this case.

Proposition 8 *If the income distribution is such that perfect sorting is optimal for the monopolist, total welfare and monopolist profits benefit equally from an increase in inequality in the form of a mean-preserving spread.*

Proof. See above. ■

The conflict between monopolist profits and welfare is thus not further intensified as inequality increases within the class of distributions for which perfect sorting is optimal for the monopolist: From a welfare perspective, no sorting would always be preferred to perfect sorting, but as inequality increases, both welfare and profits increase equally.

Importantly, the above result applies to small (infinitesimal) increases in inequality, such that perfect sorting still remains optimal for the monopolist. If the shape of the distribution changes too much, perfect sorting might no longer be the optimal sorting schedule for the monopolist. For instance, it can be shown in simulations that in case of the lognormal distribution, the function $h(.)$ is everywhere increasing in y for small σ (below 1), and hence perfect sorting is optimal for the monopolist. However, as σ increases further, there is an increasing interval of Y for which h is decreasing, which implies that pooling some regions of Y is optimal for the monopolist. In that case, welfare and profits don't necessarily increase in tandem as inequality increases.

What if the income distribution is such that perfect sorting is not optimal for the monopolist, and she therefore implements a different sorting schedule? The following proposition shows that perfect sorting is always better for total welfare than any other sorting schedule that the monopolist would design.

Proposition 9 *If the income distribution is such that the monopolist doesn't want to implement perfect sorting, a utilitarian social planner would always prefer perfect sorting to the monopolist's sorting schedule. With her optimal sorting schedule, the monopolist can rake more than half of the total surplus from sorting.*

Proof. Total surplus (i.e. the sum of all utilities) is always maximized with perfect sorting, due to supermodularity of the utility function (see Becker (1974)): Pooling everybody yields a total surplus of $\bar{y}^2$ while perfect sorting yields $E(y^2)$, which is always larger because $E(y^2) = Var(y) + \bar{y}^2$. The same holds for pooling intervals of y. As total surplus is maximized with perfect sorting, anything else must yield either the same surplus or less. With perfect sorting, citizens and the monopolist share the surplus equally. If the monopolist decides that she would rather not offer perfect sorting, it means she must expect a higher surplus with another sorting schedule, which must mean that the citizens get less than half of total surplus (and that total surplus might even be lower than that of perfect sorting). Hence, perfect sorting is always better for total welfare than any other sorting schedule that the monopolist would design. ■

To conclude, the conflict between monopolist profits and total welfare is multifaceted in the case of multiple groups: If the distribution exhibits IFR (which implies that $CV \leq 1$ and thus inequality is low), total welfare is maximized with random matching, while the monopolist wants perfect sorting, but the conflict doesn't intensify with inequality: As inequality increases (in the form of a mean-preserving spread) but we stay within the class of distributions such that the monopolist wants perfect sorting (IFR is a sufficient condition for that), welfare and monopolists profits increase equally. Starting from a situation where perfect sorting is optimal for the monopolist and inequality increases such that the monopolist wants to implement a different sorting schedule (and pool some intervals of Y), monopolist profits will increase by more and welfare will increase by less than if sorting would still be perfect.

The question of what happens to profits and total welfare if we already start from a situation where perfect sorting is not optimal for the monopolist and then see an increase in inequality is left open for future research.

6 Discussion and conclusion

This paper discusses how changes in inequality affect socio-economic segregation and resulting welfare in society. A simple two-group model is employed to show that a rise in inequality always increases profits of a monopolist who offers the sorting technology and thus also makes the supply of segregation profitable in the presence of fixed costs. Through this channel, an increase in inequality can lead to a (more) segregated society. Corresponding welfare in society, however, doesnt typically increase in line with profits. In particular when inequality is high, a conflict between welfare and profits arises and intensifies as inequality increases, and welfare decreases with inequality if the monopolist implements sorting to maximize her profits.

The paper also analyzes how sorting affects different parts of society. In particular, while sorting always lowers welfare in the poor group, welfare among the rich can be higher under sorting compared to random matching. Depending on the shape of the income distribution, the benefits of the rich can compensate for the losses of the poor, such that sorting yields higher total welfare than no sorting. However, I show that this is not necessarily the case and that for some income distributions even the rich would be better off under random matching. The intuition is that they may have to pay very high sorting fees to separate themselves off from the poor, eating up most of their utility gains from interacting only with other rich people. In those cases, sorting is thus a particularly bad idea from a welfare perspective, as nobody in society benefits from it.

The last section of the paper discusses how these findings generalize if the monopolist is not restricted to offer only one group. If the income distribution is such that perfect sorting is optimal for the monopolist initially, the prediction is clear: there is a conflict between total welfare and profits, because no sorting would be welfare maximizing. The conflict doesn't intensify for small increases in inequality, as long as perfect sorting remains optimal, but the monopolist is able to capture more than half of the total surplus if pooling for some income intervals becomes optimal. The case where perfect sorting is not optimal for the monopolist to begin with remains to be explored in future research.

This paper provides a first tentative analysis of how the welfare benefits of segregation vary with inequality if a monopolist offers the opportunity to segregate. I find that the conflict between profits and social welfare from sorting intensifies as inequality increases. While caution should be applied in translating these theoretical results into policy recommendations, it generally seems to be the case that an investigation into the benefits and losses of segregation is more relevant the higher the prevailing rate of inequality. This is due to two reasons. First, the likelihood that segregation is provided increases with inequality, and second, the analysis shows that the optimal type of segregation from a profit-maximizing and from a welfare-maximizing perspective tend to diverge as inequality rises. On a related note, the analysis in this paper can also be used to address the question as to whether, from a social welfare perspective, the government (i.e. the social planner) wants to support the provision of segregation by a monopolist or try to actively counteract it (for instance via offering social housing or employing rent support or rent control policies to prevent gentrification).

A Appendix

A.1 Sorting equilibria

Lemma 1 *In any sorting equilibrium, group S_b will have higher average income than group S_0.*

Proof. This immediately follows from 2 and from the fact that $b > 0$. ■

Lemma 2 *All sorting equilibria will be monotone.*[14] [15]

Proof. Suppose w.l.o.g. that a sorting equilibrium exists where $y_2 \in S_0$ and $y_1 \in S_b$, and $y_2 > y_1$. Then we must have

$$y_2 E[y|y \in S_b]) - y_2 E[y|y \in S_0] \leq b$$

and

$$y_1 E[y|y \in S_b] - y_1 E[y|y \in S_0] \geq b$$

and hence

$$y_1 E[y|y \in S_b] - y_1 E[y|y \in S_0] \geq y_2 E[y|y \in S_b] - y_2 E[y|y \in S_0].$$

But given that $E[y|y \in S_b] > E[y|y \in S_0]$ (see Proposition 1), this is a contradiction to $y_2 > y_1$. ■

Proposition 1 *Any sorting equilibrium is characterized by a cutoff $\hat{y} \in Y$ and a sorting fee $b > 0$ such that*

$$\hat{y}\bar{E}(\hat{y}) - \hat{y}\underline{E}(\hat{y}) = b \tag{3}$$

Proof. (Note that, as in the main part, for simplicity of notation average income in the rich group, $E[y|y \in S_b]$, is denoted by $\bar{E}(\hat{y})$ and average income in the poor group, $E[y|y \in S_0]$, by $\underline{E}(\hat{y})$.)

[14] By *monotone* I mean that the groups S_0 and S_b are single intervals of Y.

[15] If $b = 0$, then there exist trivial non-monotone sorting equilibria where the average income in both groups is the same, so that people are indifferent about which of these groups to join. Those cases are excluded by requiring that $b > 0$.

Given the equilibrium conditions, it follows that both

$$yE[y|y \in S_b] - yE[y|y \in S_0] \leq b \quad \forall y \in [0, \hat{y}]$$

and

$$yE[y|y \in S_b] - yE[y|y \in S_0] \geq b \quad \forall y \in [\hat{y}, \nu]$$

need to hold in any sorting equilibrium. This implies that a person with income $\hat{y}$ just at the border of the two groups has to be exactly indifferent between joining either of the two groups in equilibrium. Hence, we get

$$\hat{y}E[y|y \in S_b] - \hat{y}E[y|y \in S_0] = b.$$

■

A.2 Calculations for Section 5

A.2.1 Proof that for the atoms distribution no sorting is more efficient than perfect sorting, i.e. that it has $CV \leq 1$

With perfect sorting, everybody is matched only with people who earn the same income than them. The sorting fee at every income level y has to be chosen to ensure incentive compatibility, i.e. to make sure nobody would prefer to be matched with people of a different income (and pay their sorting fee). Hence, we need

$$\begin{aligned} y^2 - b(y) &\geq yy' - b(y') \\ y'^2 - b(y') &\geq y'y - b(y) \qquad \forall y, y' \in Y \end{aligned}$$

Combining the two and dividing by $y - y'$ (assuming w.l.o.g. that $y > y'$) yields

$$\frac{y'y - y'^2}{y - y'} \leq \frac{b(y) - b(y')}{y - y'} \leq \frac{y^2 - yy'}{y - y'}$$

Letting $y' \to y$ yields $b'(y) = y$. Using the fact that $b(0)$ must be 0 we get that $b(y) = \int_0^y zd(z) = \frac{y^2}{2}$. Hence, welfare from perfect sorting is given by

$$\int_0^{\nu}(y^2 - \frac{y^2}{2})f(y)dy = \int_0^{\nu}\frac{y^2}{2}f(y)dy = \frac{E(y^2)}{2}$$

Note that, as the monopolist's optimal sorting fee under perfect sorting amounts to $b(y) = \frac{y^2}{2}$, her revenue from offering perfect sorting amounts to $\int_0^{\nu}\frac{y^2}{2}f(y)dy = \frac{E(y^2)}{2}$ and the total surplus from perfect sorting $\int_0^{\nu} y^2 f(y)dy = E(y^2)$ is split equally between the monopolist and welfare.

For the atoms distribution on $[0, 1]$ we can calculate that

$$E(y^2) = \int_0^1 y^2(1-2z)dy + z = \frac{1}{3} + \frac{z}{3}$$

Therefore we see that, as described in Section 5, $E(y^2)$ (the total surplus of perfect sorting) and total welfare from perfect sorting (which is just half of it) are increasing in inequality z. However, welfare from perfect sorting is smaller than welfare from random matching for all z:

$$\frac{E(y^2)}{2} = \frac{1}{6} + \frac{z}{6} \leq \frac{1}{4} \iff z \leq 0.5$$

Another way to see this is to calculate the coefficient of variation:

$$CV = \frac{\sqrt{Var(y)}}{\bar{y}} = \frac{\sqrt{\frac{1}{3} + \frac{z}{3} - \frac{1}{4}}}{\frac{1}{2}} = 2\sqrt{\frac{1}{12} + \frac{z}{3}}$$

It is straightforward to see that $CV \leq 1\ \forall z \in [0, 0.5]$ and that it reaches its maximum of 1 where $z = 0.5$. If $z = 0.5$, perfect sorting would yield the same welfare than no sorting if the sorting fee is set at $\frac{1}{2}$, such that the total surplus is split in half. However, the sorting fee is not uniquely determined in this case and a profit-maximizing monopolist would set it as high as possible, which would be 1 in this case, such that total welfare is 0 and the monopolist gets all the surplus from sorting (which is 0.5) for herself.

A.2.2 Proof that the distributions analyzed in Section 4 are NBUE

In order to prove that the distributions analyzed in Section 4 are NBUE, we need to show that

$$\bar{E} - \bar{y} - \hat{y} < 0 \quad \forall \hat{y}, \ \ \forall z \in [-2, 2].$$

The general specification of the pdf of these distributions that encompasses all of them is

$$\begin{aligned} f(y) &= x - \frac{2z}{\nu}y \qquad \text{if } y \in \left[0, \frac{\nu}{2}\right] \\ f(y) &= x - 2z + \frac{2z}{\nu}y \quad \text{if } y \in \left[\frac{\nu}{2}, \nu\right] \end{aligned}$$

where

$$x = \frac{1}{\nu} + \frac{z}{2} \tag{8}$$

in order for $F(\nu) = 1$ and

$$z \in \left[-\frac{2}{\nu}, \frac{2}{\nu}\right]$$

which implies $x \in \left[0, \frac{2}{\nu}\right]$.

If $\hat{y} \leq 0.5$ we have that

$$\bar{E} - \bar{y} - \hat{y} = \frac{6 - 6\hat{y}^2 - 3z\hat{y}^2 + 8z\hat{y}^3}{12 - 12\hat{y} - 6z\hat{y} + 12z\hat{y}^2} - \frac{1}{2} - \hat{y}$$

$$= \frac{-6\hat{y} + 6\hat{y}^2 + z(-4\hat{y}^3 + 3\hat{y} - 3\hat{y}^2)}{12 - 12\hat{y} - 6z\hat{y} + 12z\hat{y}^2}.$$

The denominator is always positive, so we just need to analyze the numerator: $-6\hat{y} + 6\hat{y}^2$ is always negative, and $-4\hat{y}^3 + 3\hat{y} - 3\hat{y}^2$ is positive for $\hat{y} \leq 0.5$, hence if z is negative, the whole expression is negative for sure. If z is positive, then the numerator reaches its maximum at $z = 2$, where it becomes $-6\hat{y} + 6\hat{y}^2 - 8\hat{y}^3 + 6\hat{y} - 6\hat{y}^2 = -8\hat{y}^3$ which is always negative. Hence, $\bar{E} - \bar{y} - \hat{y} < 0$ if $\hat{y} \leq 0.5$.

If $\hat{y} \geq 0.5$ we have that

$$\bar{E} - \bar{y} - \hat{y} = \frac{6 - z - 6\hat{y}^2 + 9z\hat{y}^2 - 8z\hat{y}^3}{12 - 6z - 12\hat{y} + 18z\hat{y} - 12z\hat{y}^2} - \frac{1}{2} - \hat{y}$$

$$= \frac{-6\hat{y} + 6\hat{y}^2 + z(2 + 4\hat{y}^3 - 3\hat{y} - 3\hat{y}^2)}{12 - 6z - 12\hat{y} + 18z\hat{y} - 12z\hat{y}^2}.$$

The denominator is again positive, and the first term of the numerator, $-6\hat{y} + 6\hat{y}^2$ is always negative. $2 + 4\hat{y}^3 - 3\hat{y} - 3\hat{y}^2$ reaches its minimum at $\frac{1}{4} + \sqrt{\frac{5}{16}}$ where it is negative, and hence $z(2 + 4\hat{y}^3 - 3\hat{y} - 3\hat{y}^2)$ is positive if $z < 0$, and maximal at $z = -2$. Combined with $-6\hat{y} + 6y^2$ evaluated at $\frac{1}{4} + \sqrt{\frac{5}{16}}$ the total expression is negative. $2 + 4\hat{y}^3 - 3\hat{y} - 3\hat{y}^2$ reaches its maximum at 0.5 where it is positive and hence $z(2 + 4\hat{y}^3 - 3\hat{y} - 3\hat{y}^2)$ is maximal at $z = 2$. Again combined with $-6\hat{y} + 6y^2$ evaluated at 0.5 the whole expression is negative. Hence $\bar{E} - \bar{y} - \hat{y} < 0$ if $\hat{y} > 0.5$, and thus the examined distribution is NBUE for all z.

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ONLINE APPENDIX
for “Monopolistic Supply of Sorting, Inequality and Welfare”

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Contents

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1 Symmetric atoms distribution

This section examines a stylized income distribution which I call *symmetric atoms distribution*. For this distribution, I will calculate the monopolist's profit-maximizing cutoff. I will also explore which cutoff would maximize total welfare (or whether random matching would be more efficient) and compare it to total welfare from sorting at the monopolist's optimal cutoff. Similar analyses will be conducted for welfare of the richest individuals and average welfare of the poor. Furthermore, I will explore how the monopolist's optimal cutoff changes with inequality, and also examine how total welfare, welfare of the richest individuals and average welfare in the poor group from sorting at the monopolist's optimal cutoff are affected.

As the below calculations show, it turns out that sorting at the monopolist's optimal cutoff is never beneficial for total welfare in case of the symmetric atoms distribution, and that even the richest individuals would prefer random matching to the monopolist's sorting schedule. Furthermore, total welfare from sorting (at the monopolist's optimal cutoff) decreases in inequality, and the same holds for average welfare in the poor group and welfare of the richest individuals. As monopolist profits increase with inequality (see Proposition 2), an increase in inequality thus increases the conflict between monopolist profits and welfare.

The symmetric atoms distribution is characterized by two atoms at 0 and ν, each with mass z (and is uniformly distributed in between). This distribution is of course not usually encountered in reality. However, it is easy to handle and - despite its stylized shape - can be deployed to analyze (via varying parameter z) the implications of a society that is "drifting apart", where the rich are getting richer and the poor are becoming poorer. Average income is $\bar{y} = \frac{\nu}{2}$, irrespective of z, and the conditional expectations are

$$\underline{E}(\hat{y}) = \frac{\left(\frac{1-2z}{\nu}\right)\frac{\hat{y}^2}{2}}{z + \left(\frac{1-2z}{\nu}\right)\hat{y}}$$

and

$$\bar{E}(\hat{y}) = \frac{z\nu + \left(\frac{1-2z}{\nu}\right)\left(\frac{\nu^2}{2} - \frac{\hat{y}^2}{2}\right)}{z + \left(\frac{1-2z}{\nu}\right)(\nu - \hat{y})}.$$

Note that z must be in the interval $[0, 0.5]$ and that $z = 0$ implies that income is uniformly distributed. Furthermore, z parameterizes inequality (in the sense of the difference $\bar{E}(\hat{y}) - \underline{E}(\hat{y})$ for any cutoff), and an increase in z is a monotone mean-preserving spread of the income distribution (and therefore implies an increase in the Gini-coefficient of the distribution).

1.1 Profit-maximizing cutoff

In order to explore the implications of increasing inequality for total welfare from sorting at the monopolist's optimal cutoff, we first need to examine how this optimal cutoff varies with inequality. From Proposition 2 we know that the monopolist's profit is increasing in inequality (and thus in z). In order to identify how the monopolist's optimal cutoff is affected by an increase in inequality, the following Lemma is derived:

Lemma A1 *If the income distribution is continuously differentiable and such that it can be written as $F(y, z)$, where z parameterizes inequality and an increase in z is a monotone mean-preserving spread of the income distribution, then an increase in z increases the monopolist's profit-maximizing cutoff if the income distribution is such that*

$$\frac{\partial^2 \underline{E}(\hat{y}^*, z)}{\partial \hat{y} \partial z} \leq 0 \quad \text{and} \quad \frac{\partial^2 \underline{E}(\hat{y}^*, z)}{(\partial \hat{y})^2} \geq 0.$$

Proof. If the monopolist's revenue function is strictly concave, such that her maximization problem has a unique solution, the monopolist's optimal cutoff is characterized via the first order condition

$$\frac{dR(\hat{y}^*, z)}{d\hat{y}} = 0.$$

The monopolist's revenue is

$$R(\hat{y}, z) = \hat{y}(\bar{y} - \underline{E}(\hat{y}, z))$$

and the optimal cutoff is thus given by

$$\bar{y} - \underline{E}(\hat{y}^*, z) = \hat{y}^* \frac{\partial \underline{E}(\hat{y}^*, z)}{\partial \hat{y}}.$$

Taking the derivative with respect to z gives

$$-\frac{\partial \underline{E}(\hat{y}^*, z)}{\partial \hat{y}}\frac{d\hat{y}^*}{dz}-\frac{\partial \underline{E}(\hat{y}^*, z)}{\partial z}=\hat{y}\frac{\partial^2 \underline{E}(\hat{y}^*, z)}{(\partial \hat{y})^2}\frac{d\hat{y}^*}{dz}+\frac{\partial \underline{E}(\hat{y}^*, z)}{\partial \hat{y}}\frac{d\hat{y}^*}{dz}+\hat{y}\frac{\partial^2 \underline{E}(\hat{y}^*, z)}{\partial \hat{y}\partial z}$$

and therefore

$$\frac{-\frac{\partial \underline{E}(\hat{y}^*,z)}{\partial z}-\hat{y}^*\frac{\partial^2 \underline{E}(\hat{y}^*,z)}{\partial \hat{y}\partial z}}{\hat{y}^*\frac{\partial^2 \underline{E}(\hat{y}^*,z)}{(\partial \hat{y})^2}+2\frac{\partial \underline{E}(\hat{y}^*,z)}{\partial \hat{y}}}=\frac{d\hat{y}^*}{dz}.$$

Because an increase in z is a monotone mean-preserving spread, we have that $\frac{\partial \underline{E}(\hat{y},z)}{\partial z}<0$. Furthermore, an increase in the cutoff always increases average income below the cutoff, therefore $\frac{\partial \underline{E}(\hat{y},z)}{\partial \hat{y}}>0$. Sufficient conditions for

$$\frac{d\hat{y}^*}{dz}>0$$

are therefore

$$\frac{\partial^2 \underline{E}(\hat{y}^*, z)}{\partial \hat{y}\partial z}\leq 0 \quad \text{and} \quad \frac{\partial^2 \underline{E}(\hat{y}^*, z)}{(\partial \hat{y})^2}\geq 0.$$

The monopolist's profit maximization problem is guaranteed to have a unique solution if the revenue function is strictly concave in $\hat{y}$, i.e. $\frac{\partial^2 R(\hat{y},z)}{(\partial \hat{y})^2}<0$ for all $\hat{y}$. We have that

$$\frac{\partial^2 R(\hat{y}, z)}{(\partial \hat{y})^2}=-2\frac{\partial \underline{E}(\hat{y}, z)}{\partial \hat{y}}-\hat{y}\;\frac{\partial^2 \underline{E}(\hat{y}, z)}{(\partial \hat{y})^2}.$$

$\frac{\partial \underline{E}(\hat{y},z)}{\partial \hat{y}}$ is always positive, hence the whole expression is negative for sure if $\frac{\partial^2 \underline{E}(\hat{y},z)}{(\partial \hat{y})^2}\geq 0$, which is exactly one of the sufficient conditions above. Hence, this condition ensures both that the monopolist's optimal cutoff is unique and (together with the condition for the cross derivative) that this optimal cutoff increases with inequality. ■

The above proof shows that the sufficient conditions from Lemma A1 hold for the symmetric atoms distribution on the interval $(0, \nu)$. Hence, the monopolist's revenue function is strictly concave on that interval and there is a unique optimal cutoff, which is increasing in z. On the entire interval $[0, \nu]$ there are three candidates for a global optimum for the monopolist: 0, ν and the unique interior optimum. It is obvious that 0 cannot be the optimum, because the monopolist's revenue would be zero in that case.

Proposition A1 shows that the interior optimum is the monopolist's global optimal cutoff for all $z \in [0, 0.5]$ and that the revenue obtained from it coincides with the revenue from cutoff ν if $z = 0.5$. We can thus conclude that:

Proposition A1 *The monopolist's optimal cutoff is increasing in z. For $z = 0$ the optimal cutoff is at $\frac{\nu}{2}$. Hence, the monopolist's optimal cutoff is located in the interval $[\frac{\nu}{2}, \nu]$ for all z and characterized via the first order condition $\frac{dR(\hat{y}^*,z)}{d\hat{y}} = 0$.*

Proof. As Lemma A1 establishes, sufficient conditions for

$$\frac{d\hat{y}^*}{dz} > 0$$

are

$$\frac{\partial^2 \underline{E}(\hat{y}^*, z)}{\partial \hat{y} \partial z} < 0 \quad \text{and} \quad \frac{\partial^2 \underline{E}(\hat{y}^*, z)}{(\partial \hat{y})^2} > 0.$$

Show that $\frac{\partial^2 \underline{E}(\hat{y},z)}{\partial \hat{y} \partial z} < 0$: (Note: To simplify the notation $\nu = 1$ in the following calculations, but everything works analogously if the distribution is scaled up to a general $\nu > 0$).

$$\underline{E}(\hat{y}, z) = \frac{(1-2z)\frac{\hat{y}^2}{2}}{z + (1-2z)\hat{y}}$$

$$\begin{aligned} \frac{\partial \underline{E}(\hat{y}, z)}{\partial \hat{y}} &= \frac{(1-2z)\hat{y}(z + (1-2z)\hat{y}) - (1-2z)^2 \frac{\hat{y}^2}{2}}{(z + (1-2z)\hat{y})^2} \\ &= \frac{(1-2z)}{z + (1-2z)\hat{y}} \left(\hat{y} - \frac{(1-2z)\frac{\hat{y}^2}{2}}{z + (1-2z)\hat{y}} \right) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \underline{E}(\hat{y}, z)}{\partial \hat{y} \partial z} &= \frac{-2(z + (1-2z)\hat{y}) - (1-2z)(1-2\hat{y})}{(z + (1-2z)\hat{y})^2} \left(\hat{y} - \frac{(1-2z)\frac{\hat{y}^2}{2}}{z + (1-2z)\hat{y}} \right) \\ &\quad - \frac{(1-2z)}{z + (1-2z)\hat{y}} \left(\frac{\hat{y}^2}{2} \left(\frac{-2(z + (1-2z)\hat{y}) - (1-2z)(1-2\hat{y})}{(z + (1-2z)\hat{y})^2} \right) \right) \\ &= \frac{-1}{(z + (1-2z)\hat{y})^2} \left(\hat{y} - \frac{(1-2z)\hat{y}^2}{z + (1-2z)\hat{y}} \right) \end{aligned}$$

$$= \frac{-1}{(z+(1-2z)\hat{y})^2}\left(\frac{z+(1-2z)\hat{y}-(1-2z)\hat{y}}{z+(1-2z)\hat{y}}\right)\hat{y}$$

$$= \frac{-1}{(z+(1-2z)\hat{y})^2}\left(\frac{z}{z+(1-2z)\hat{y}}\right)\hat{y} < 0$$

Show that $\frac{\partial^2 \underline{E}(\hat{y},z)}{(\partial \hat{y})^2} > 0$:

$$\frac{\partial^2 \underline{E}(\hat{y},z)}{(\partial \hat{y})^2} = -\frac{(1-2z)^2}{(z+(1-2z)\hat{y})^2}\left(\hat{y}-\frac{(1-2z)\frac{\hat{y}^2}{2}}{z+(1-2z)\hat{y}}\right) + \frac{(1-2z)}{z+(1-2z)\hat{y}}\left(1+\frac{(1-2z)^2}{(z+(1-2z)\hat{y})^2}\frac{\hat{y}^2}{2}-\frac{(1-2z)\hat{y}}{z+(1-2z)\hat{y}}\right)$$

$$= \frac{(1-2z)}{z+(1-2z)\hat{y}}\left(1-\frac{(1-2z)\hat{y}}{z+(1-2z)\hat{y}}\right)^2 = \frac{(1-2z)}{z+(1-2z)\hat{y}}\left(\frac{z}{z+(1-2z)\hat{y}}\right)^2 > 0$$

The above implies that the monopolist's revenue is strictly concave on $(0,\nu)$ and thus has a unique optimum on this interval. Hence, there are three candidates for a global optimum: 0, ν and the interior optimum. It is obvious that 0 cannot be the optimum, because the monopolist's revenue would be zero. At ν, the monopolist's revenue would be

$$R(1) = 1.z.(1-\underline{E}(1)) = \frac{z}{2-2z}$$

It can be shown (numerically) that $R(1)$ is smaller than the interior optimum for any $z \in [0, 0.5]$. If $z = 0$, then

$$\underline{E}(\hat{y}) = \frac{\hat{y}}{2}$$

and therefore

$$R(\hat{y}) = \hat{y}\left(\frac{\nu}{2}-\frac{\hat{y}}{2}\right) = \frac{\hat{y}\nu}{2}-\frac{\hat{y}^2}{2}$$

This implies that profit is maximized at $\frac{\nu}{2}$ for $z = 0$. ■

1.2 Welfare

What happens to welfare from sorting at the monopolist's optimal cutoff as inequality increases? In order to explore this, Proposition A2 first examines whether or not sorting would be optimal for total welfare if the cutoff could be chosen optimally (i.e. to maximize welfare). Total welfare without sorting is independent of inequality, it is $\bar{y}^2 = \frac{\nu^2}{4}$ for all z. For strictly positive z, Proposition A2 finds that sorting at small but positive $\hat{y}$ yields higher total welfare than no sorting, but sorting at the monopolist's optimal cutoff (which, as Proposition A1 shows, is always greater than $\frac{\nu}{2}$) is always less efficient than no sorting. Total welfare is always highest at $\hat{y} = 0$, i.e. if everybody except the mass of people with 0 income is in the rich group.

Proposition A2 1. *If $z = 0$ (uniform distribution), maximal welfare is achieved with no sorting.*

2a. *If $z > 0$, maximum welfare is attained at $\hat{y} = 0$ for all z, i.e. it is optimal for the rich group to consist of everybody except people with 0 income. Furthermore, welfare of sorting at $\hat{y} = 0$ is increasing in z.*

2b. *If $z > 0$, there is a range of $\hat{y} \geq 0$ for which sorting at these $\hat{y}$ yields higher welfare than no sorting. This range increases with z and becomes $[0, \frac{\nu}{2})$ if $z = 0.5$. No sorting is therefore always more efficient than sorting at the monopolist's optimal cutoff (which is always above $\frac{\nu}{2}$).*

Proof. First of all, note that for strictly positive z welfare jumps at 0: If people with 0 income are included in the group (so there is only one group), total welfare is $\frac{\nu^2}{4}$, if 0 is excluded, then welfare is $\frac{\nu^2}{4-4z} > \frac{\nu^2}{4}$. This comes from the fact that if people with 0 are excluded and everybody with income greater than zero is in the rich group, then the corresponding sorting fee is 0. Hence, the equilibrium where people with 0 income are excluded from the rich group necessarily yields higher welfare, because the rich group benefits from not having to interact with the zero income population, and the people with zero income are indifferent, because their utility is zero in any case.

For the remainder of this proof I will again set $\nu = 1$ for simplicity of notation. Welfare at cutoff $\hat{y}$ is then given by

$$\begin{aligned} W(\hat{y}) &= (z+(1-2z)\hat{y})\left(\frac{(1-2z)\frac{\hat{y}^2}{2}}{(z+(1-2z)\hat{y})}\right)^2 \\ &\quad +(z+(1-2z)(1-\hat{y}))\left(\frac{z+(1-2z)\left(\frac{1}{2}-\frac{\hat{y}^2}{2}\right)}{z+(1-2z)(1-\hat{y})}\right)^2 \\ &\quad -\hat{y}\left(\frac{1}{2}-\frac{(1-2z)\frac{\hat{y}^2}{2}}{z+(1-2z)\hat{y}}\right) \end{aligned}$$

If $z=0$ the distribution becomes uniform and welfare is

$$W_{z=0}(\hat{y}) = \hat{y}\frac{\hat{y}^2}{4} + (1-\hat{y})\left(\frac{\frac{1}{2}(1-\hat{y}^2)}{(1-\hat{y})}\right)^2 - \hat{y}\left(\frac{1}{2}-\frac{\hat{y}}{2}\right) = \frac{1}{4}-\frac{\hat{y}}{4}+\frac{\hat{y}^2}{4}$$

It is straightforward to see that this quadratic function reaches its minimum at $\hat{y}=0.5$ and is maximized at the endpoints of the examined interval Y, i.e. $\hat{y}=0$ and $\hat{y}=1$, where welfare is $\frac{1}{4}$, which is equal to the total welfare of no sorting. Hence, no sorting is (weakly) preferred to sorting at any cutoff $\hat{y} \in Y$ if $z=0$.

For the general case, where $z \neq 0$, note first that total welfare at cutoff 0 is

$$W(0) = \frac{\left(z+(1-2z)\frac{1}{2}\right)^2}{z+(1-2z)} = \frac{1}{4(1-z)}$$

which is increasing in z for all $z \in [0, 0.5]$. It is also straightforward to see that this expression is always larger than $\frac{1}{4}$ (welfare of no sorting) if $z>0$. It can be shown (numerically) that welfare from sorting is maximized at $\hat{y}=0$ if $z>0$.

At cutoff 1 welfare from sorting becomes

$$W(1) = \frac{1-2z}{4(1-z)}$$

which is decreasing in z for all $z \in [0, 0.5]$. Note that this is always smaller than $\frac{1}{4}$ (welfare of no sorting) for all $z>0$.

For all cutoffs in between 0 and 1, note that from the previous section we know that sorting yields higher welfare than no sorting at cutoff $\hat{y}$ iff

$$\bar{E} - \bar{y} - \hat{y} > 0$$

Plugging in the expressions for $\bar{E}$ and $\bar{y}$ for the atoms distribution, this condition becomes

$$\frac{(1-2z)\hat{y}^2 - \hat{y} + z}{2(z + (1-2z)(1-\hat{y}))} > 0.$$

As the numerator of this fraction is positive for all z and $\hat{y}$, the condition can be simplified to

$$(1-2z)\hat{y}^2 - \hat{y} + z > 0$$

It is immediate to see that this condition never holds if $z = 0$, holds for all $z > 0$ at cutoff 0, and holds for all $\hat{y} \leq 0.5$ if $z = 0.5$. The roots of $(1-2z)\hat{y}^2 - \hat{y} + z$ are

$$y_{1,2} = \frac{1 \pm \sqrt{1 - 4z + 8z^2}}{2 - 4z}$$

and the polynomial is positive for all $\hat{y}$ that are either smaller than the smaller of the two or larger than the larger of the two roots. As the larger root is always ≥ 1, the only relevant case for us is the range of $\hat{y}$ smaller than $y_1 = \frac{1-\sqrt{1-4z+8z^2}}{2-4z}$. The value of y_1 is 0 if $z = 0$ and is then increasing in z, until it reaches $y_1 = 0.5$ for $z = 0.5$. Hence, the range of $\hat{y}$ for which sorting is better than no sorting is $[0, y_1(z)]$ for all $z > 0$ where $y_1(z)$ is increasing in z, 0 for $z = 0$ and reaches 0.5 for $z = 0.5$. ■

As Points 2a and 2b in the above proposition indicate, total welfare from sorting is higher than welfare from random matching if the cutoff is chosen to maximize welfare. The welfare-maximizing partition for all positive z is such that everybody except for people with zero income are part of the rich group, and welfare of those people (which is equal to total welfare, because welfare of people in the poor group is zero) increases in inequality. However, note that this result is very specific to the symmetric atoms distribution and largely due to the fact that inequality is parameterized by z, which captures the mass of poor and rich people in society. As z increases, the mass of maximally rich people with whom the rich group interacts increases, thus pushing up everybody's utility. That at the same time there are more people who have zero income and live isolated from the rest of society doesn't matter for total welfare, because people with zero income would achieve zero utility irrespective of the sorting regime. It is thus necessarily beneficial in terms of total welfare to separate the very poor from the rest of society. This is clearly a very artificial result specific to the particular distribution and should not be taken at face value.[1]

The fact that total welfare is higher under sorting (if the cutoff is chosen appropriately) compared to random matching and even increases with inequality is very specific to the symmetric atoms distribution and not found for all the other distributions examined in this paper. (The other distributions are all NBUE, which implies that sorting can never yield higher welfare than random matching.) However, what *is* common to all distributions examined in this paper is that there is in general a conflict between welfare and profit maximization. For the symmetric atoms distribution, Proposition A2 states that total welfare is maximized if sorting is at cutoff $\hat{y} = 0$. Clearly, no profit-maximizing monopolist would choose this cutoff, because the revenue at this cutoff would be 0 (because the sorting fee would be 0). On the other hand, no sorting always yields higher welfare than sorting at the monopolist's optimal cutoff, and a profit-maximizing monopolist would never choose a cutoff that yields higher welfare than no sorting. The following proposition shows that this conflict increases with inequality:

Proposition A3 *Total welfare at the monopolist's optimum is decreasing in z if z is large enough.*

Proof. The derivative of welfare with respect to z at the monopolist's optimal cutoff $\hat{y}^*$ amounts to

$$\frac{dW(\hat{y}^*, z)}{dz} = \left(f(\underline{E}^2 - \bar{E}^2) + F2\underline{E}\frac{\partial \underline{E}}{\partial \hat{y}} + (1-F)2\bar{E}\frac{\partial \bar{E}}{\partial \hat{y}} \right) \frac{d\hat{y}^*}{dz}$$

$$+\frac{\partial F}{\partial z}(\underline{E}^2 - \bar{E}^2) + F2\underline{E}\frac{\partial \underline{E}}{\partial z} + (1-F)2\bar{E}\frac{\partial \bar{E}}{\partial z} - \frac{d\Pi(\hat{y}^*, z)}{dz}$$

[1] Besides, any government or social planner (openly) trying to segregate off the poorest people such that the rest of society can reap higher utility because they don't have to interact with the very poor would have trouble justifying their regime, even if it maximizes total welfare.

where $\Pi(\hat{y}^*, z)$ is the monopolist's maximized profit. We know that the monopolist's profit maximization always has an interior solution (see Lemma A1 and Proposition A1). Hence the optimal cutoff y^* is characterized via the first order condition

$$\frac{\partial \Pi(\hat{y}^*, z)}{\partial \hat{y}} = 0$$

This implies that

$$\frac{d\Pi(\hat{y}^*, z)}{dz} = \frac{\partial \Pi(\hat{y}^*, z)}{\partial \hat{y}}\frac{d\hat{y}^*}{dz} - \hat{y}^*\frac{\partial \underline{E}}{\partial z} = -\hat{y}^*\frac{\partial \underline{E}}{\partial z}(> 0).$$

Hence, the above expression can be simplified to

$$\frac{dW(\hat{y}^*, z)}{dz} = f(\bar{E} - \underline{E})(\bar{E} + \underline{E} - 2\hat{y}^*)\frac{d\hat{y}^*}{dz} +$$

$$+\frac{\partial F}{\partial z}(\underline{E}^2 - \bar{E}^2) + (F2\underline{E} + \hat{y}^*)\frac{\partial \underline{E}}{\partial z} + (1 - F)2\bar{E}\frac{\partial \bar{E}}{\partial z}$$

(where I also use the fact that $\frac{\partial \underline{E}(\hat{y},z)}{\partial \hat{y}} = f(\hat{y}, z)\frac{\hat{y} - \underline{E}(\hat{y},z)}{F(\hat{y},z)}$ and $\frac{\partial \bar{E}(\hat{y},z)}{\partial \hat{y}} = f(\hat{y}, z)\frac{\bar{E}(\hat{y},z) - \hat{y}}{1 - F(\hat{y},z)}$).

We have

$$\frac{\partial F}{\partial z} = 1 - 2\hat{y}$$

and

$$\frac{\partial \underline{E}}{\partial z} = \frac{-\frac{\hat{y}^2}{2}}{(z + (1 - 2z)\hat{y})^2}$$

and

$$\frac{\partial \bar{E}}{\partial z} = \frac{\frac{1}{2}(\hat{y} - 1)^2}{(z + (1 - 2z)(1 - \hat{y}))^2}.$$

Note that $\frac{\partial \underline{E}}{\partial z} < 0$, $\frac{\partial F}{\partial z} < 0$ (because $\hat{y}^* > \frac{\nu}{2}$) and $\frac{\partial \bar{E}}{\partial z} > 0$. From Proposition A1 we know that $\frac{d\hat{y}^*}{dz} > 0$, hence sufficient conditions for $\frac{dTW(\hat{y}^*,z)}{dz} < 0$ are that

$$\bar{E} + \underline{E} - 2\hat{y}^* < 0$$

and

$$\frac{\partial F}{\partial z}(\underline{E}^2 - \bar{E}^2) + (F2\underline{E} + \hat{y}^*)\frac{\partial \underline{E}}{\partial z} + (1 - F)2\bar{E}\frac{\partial \bar{E}}{\partial z} < 0$$

The first condition can easily be shown to always hold for $\hat{y}^* > 0.5$: Plugging in the expressions for $\bar{E}$ and $\underline{E}$ yields

$$\bar{E} + \underline{E} - 2\hat{y}^* = \left(\frac{z}{2} - z\hat{y}\right) + (2\hat{y} - 6\hat{y}^2 + 4\hat{y}^3)(\frac{1}{4} + z^2 - z)$$

We have that

$$\left(\frac{z}{2} - z\hat{y}\right) < 0 \quad \forall \hat{y} > 0.5$$

and

$$(2\hat{y} - 6\hat{y}^2 + 4\hat{y}^3) < 0 \quad \forall \hat{y} > 0.5$$

while

$$(\frac{1}{4} + z^2 - z) > 0 \quad \forall z > 0.5$$

Hence, the total expression is always negative for $\hat{y} > 0.5$. For the second condition, note that

$$\frac{\partial F}{\partial z}(\underline{E}^2 - \bar{E}^2) + (F2\underline{E} + \hat{y}^*)\frac{\partial \underline{E}}{\partial z} + (1-F)2\bar{E}\frac{\partial \bar{E}}{\partial z} =$$

$$= \frac{(1-2z)^2\left(\frac{\hat{y}^4}{4} - \frac{\hat{y}^5}{2}\right) - (1-2z)\frac{\hat{y}^4}{2} - \frac{\hat{y}^3}{2}}{(z + (1-2z)\hat{y})^2} + $$
$$+ \frac{z(1-y)^2 + (1-2z)\left(\frac{1}{2} - \frac{\hat{y}^2}{2}\right)(1-\hat{y})^2 - (1-2\hat{y})\left(z + (1-2z)\left(\frac{1}{2} - \frac{\hat{y}^2}{2}\right)\right)^2}{\left(z + (1-2z)(1-\hat{y})\right)^2}$$

The first summand of this expression is negative for all $\hat{y} > 0.5$ and all z, but the second term is always positive (note that $1 - 2\hat{y} < 0$ for all $\hat{y} > 0.5$). If $z = 0$ the sum of the two becomes $\frac{1}{4} - \frac{1}{2}\hat{y}$ which is negative for all $\hat{y} > 0.5$, however if $z > 0$ then there is a small range of $\hat{y} > 0.5$ for which the second term is higher in absolute value than the first and hence the whole expression is positive. Indeed it can be shown that the entire expression for $\frac{dW(\hat{y}^*,z)}{dz}$ is positive for small $\hat{y}^* > 0.5$ (from numerical simulations). As $\hat{y}^*$ is close to 0.5 for small z this implies that welfare from sorting at the monopolist's optimal cutoff increases with z for very small z. However, note that the monopolist's optimal cutoff increases with z as well, and this increase moves $\hat{y}^*$ out of the area for which welfare increases with z quickly. It can be seen (from numerical simulations) that for all $z > 0.05$ the small range of $\hat{y}$ for which welfare increases with z is below

$\hat{y}^*$ for all z. Hence, welfare from sorting at the monopolist's optimal cutoff decreases with z if $z > 0.05$. ∎

Proposition A3 thus shows that sorting (at the monopolist's optimal cutoff) becomes less socially desirable as inequality increases. In addition to analyzing total welfare, it is also instructive to examine how welfare of the richest varies with z. Clearly, sorting is never desirable for people in the poor group, as they would always be better off with random matching (though, as demonstrated below, it is also interesting to explore how average welfare in the poor group varies with inequality). The only reason why a benevolent social planner might be in favour of sorting is because it can benefit the rich. (Those benefits could then also be redistributed among the rest of society and make everybody better off, but even without redistribution a social planner who puts high weight on the benefits of the rich might support sorting). As Proposition A4 shows, the utility that the very richest in society derive from sorting gives us an upper bound on how much anybody in society benefits from sorting at some $\hat{y}$ compared to random matching:

Proposition A4 *The utility difference between sorting at some cutoff $\hat{y}$ and no sorting is increasing in y, i.e. if a person with income y prefers no sorting to sorting at some $\hat{y}$, then also everybody with income smaller than y prefers no sorting to sorting.*

Proof. Utility from sorting at $\hat{y}$ for a person with income $y \geq \hat{y}$ is

$$y\bar{E}(\hat{y}) - \hat{y}(\bar{E}(\hat{y}) - \underline{E}(\hat{y}))$$

and utility from no sorting is

$$y\bar{y},$$

hence the utility difference amounts to

$$y\bar{E}(\hat{y}) - \hat{y}(\bar{E}(\hat{y}) - \underline{E}(\hat{y})) - y\bar{y}$$

where a positive difference implies that sorting at $\hat{y}$ yields higher utility than no sorting. The derivative of this difference with respect to y (for given $\hat{y}$) is $\bar{E}(\hat{y}) - \bar{y}$ which is always positive. Hence, utility of sorting is increasing in income for members of the rich group. The people just at $\hat{y}$ - who are in the rich group - will derive utility $\hat{y}\underline{E}$ and everybody in the poor group will derive less utility and it is straightforward to see that utility in the poor group is also increasing in income. Hence, utility from sorting at cutoff $\hat{y}$ is increasing in income for everybody in the economy. ■

The following results can be derived for welfare of people with income ν (denoted by W_ν):

Proposition A5 *1. If $z = 0$, welfare of people with income ν is constant and equal to $\frac{\nu^2}{2}$, irrespective of whether there is sorting or not.*

2. If $z > 0$ then welfare of people with income ν is equal to $\frac{\nu^2}{2}$ without sorting, but it is higher than $\frac{\nu^2}{2}$ if there is sorting at any cutoff $\hat{y} \in \left[0, \frac{\nu}{2}\right)$. Hence, people with income ν prefer sorting at any $\hat{y} \in \left[0, \frac{\nu}{2}\right)$ to no sorting. However, no sorting is always preferred to sorting at $\hat{y} > \frac{\nu}{2}$.

3. W_ν at those $\hat{y}$ for which sorting is better than no sorting (i.e. all $\hat{y} < \frac{\nu}{2}$) increases with z and is highest if $z = 0.5$.

4. If $z > 0$, W_ν is maximized at $\hat{y} = 0$, i.e. when everybody except people with zero income is in the rich group. However, no sorting is always preferred to the monopolist's optimal cutoff for $z > 0$ (because the monopolist's optimal cutoff is always larger than $\frac{\nu}{2}$).

Proof. Welfare of people with income ν can be calculated as

$$W_\nu(\hat{y}, z) = \nu\bar{E} - \hat{y}(\bar{E} - \underline{E}).$$

This can be written as (again set $\nu = 1$)

$$W_\nu(\hat{y}, z) = \frac{\frac{z}{2} - \frac{3}{2}z\hat{y} + \frac{z\hat{y}^2}{2} + z\hat{y}^3 + z^2\hat{y}^2 - 2z^2\hat{y}^3 + \frac{\hat{y}}{2} - \frac{\hat{y}^2}{2}}{(z + (1 - 2z)\hat{y})(z + (1 - 2z)(1 - \hat{y}))}.$$

If $z = 0$ (uniform distribution) this becomes

$$W_\nu(\hat{y}, 0) = \frac{1}{2}$$

Note that utility of no sorting is also $\frac{1}{2}$ for people with income ν, they are therefore indifferent between sorting and no sorting at any cutoff if $z = 0$. The intuition for this result is that if the income distribution is uniform, the rich have to pay a high sorting fee to separate themselves off from the rest, and this diminishes their utility from sorting. In fact, only the richest (weakly) benefit from sorting in case of the uniform distribution, anybody in the rich group with income smaller than ν would be better off without sorting.

If $z > 0$: When is $W_\nu(\hat{y}) > 0.5$ (=utility from no sorting), i.e. for what range of cutoffs is sorting preferred to no sorting for the richest people?

$$W_\nu(\hat{y}, z) > 0.5$$

$$\iff \frac{z}{2} - \frac{3}{2}z\hat{y} + \frac{z\hat{y}^2}{2} + z\hat{y}^3 + z^2\hat{y}^2 - 2z^2\hat{y}^3 + \frac{\hat{y}}{2} - \frac{\hat{y}^2}{2} > \frac{(z + (1-2z)\hat{y})(z + (1-2z)(1-\hat{y}))}{2}$$

$$z(\hat{y} - 3\hat{y}^2 + 2\hat{y}^3) + z^2(6\hat{y}^2 - 4\hat{y}^3 - 4\hat{y} + 1) > 0$$

If $z > 0$ this becomes

$$(\hat{y} - 3\hat{y}^2 + 2\hat{y}^3) > z(-6\hat{y}^2 + 4\hat{y}^3 + 4\hat{y} - 1)$$

The RHS is positive if $\hat{y} \geq 0.5$, which is also exactly when the LHS is negative. (The polynomial on the left has roots 1 and 0.5 and is smaller than zero in between the two and larger than zero elsewhere. The polynomial on the right has only root 0.5 in the interval $[0, 1]$ and is positive above 0.5 and negative below). In other words, the inequality cannot hold for any positive z if $\hat{y} \geq 0.5$ and will always hold if $\hat{y} \leq 0.5$. This means that for any z the richest people prefer sorting to no sorting at any cutoff below 0.5. It is straightforward to see that they are always indifferent between sorting and no sorting at $\hat{y} = 0.5$. The maximum utility is reached at $\hat{y} = 0$ (meaning that the rich group consists of everybody except the poor with zero income) for any $z > 0$, which can be concluded from the fact that $\frac{dW_\nu(\hat{y},z)}{d\hat{y}} < 0$ for all $\hat{y} \in [0, 1]$.

Proof that $\frac{dW_\nu(\hat{y},z)}{d\hat{y}}$ for all $\hat{y} \in [0,1]$:

$$\frac{dW_\nu(\hat{y},z)}{d\hat{y}} = \frac{\begin{array}{c} -z^4 + (1-2z)z^3(-0.5+3\hat{y}^2-3\hat{y}) + (1-2z)^2 2z^2\hat{y}(-1+\hat{y}) \\ +(1-2z)^3 z(-\hat{y}^2-\hat{y}^4+2\hat{y}^3) \end{array}}{(z+(1-2z)\hat{y})^2(z+(1-2z)(1-\hat{y}))^2}$$

The denominator is always positive, so it suffices to focus on the numerator. Analysis of the factors that multiply the potencies of $(1-2z)$ shows that they are negative for all $\hat{y} \in [0,1]$ and hence $\frac{dW_\nu(\hat{y},z)}{d\hat{y}}$ is smaller than 0 for all $\hat{y} \in [0,1]$. The maximum welfare for the rich is therefore achieved when $\hat{y} = 0$ (i.e. the rich group consists of everybody except the poorest with income 0). This maximum welfare is increasing in z:

$$W_\nu(0,z) = \frac{1}{2(1-z)}$$

On the other hand, welfare at $\hat{y} = \nu(=1)$ is decreasing in z:

$$W_\nu(1,z) = \frac{\frac{1}{2}-z}{(1-z)}.$$

■

Proposition A5 demonstrates that the richest individuals in society don't necessarily benefit from sorting, in particular not when the sorting schedule is chosen by a profit-maximizing monopolist. For the symmetric atoms distribution it is in fact the case that the rich *never* prefer sorting at the monopolist's optimal cutoff to random matching. By Proposition A4, this implies that *nobody* in society prefers sorting at the monopolist's optimal cutoff to no sorting. Hence, the symmetric atoms distribution identifies a case where sorting at the monopolist's optimal cutoff is not only not beneficial for total welfare (i.e. the benefits of the rich cannot compensate for the losses of the poor), it is not even desirable for the richest people in society.

The next proposition demonstrates what happens to welfare of the richest individuals from sorting at the monopolist's optimal cutoff as inequality increases. An increase in inequality has two effects on the richest people in society: Their group gets richer on average (because there is more mass at the top end and because the cutoff increases) but at the same time they have to pay a higher sorting fee, because the difference between rich and poor, which determines the sorting fee, increases. Proposition A6 shows that net effect on their welfare is negative.

Proposition A6 *Welfare of the richest individuals from sorting at the monopolist's optimum is decreasing in z.*

Proof.

$$W_\nu(\hat{y}^*, z) = (\nu - \hat{y}^*)\bar{E}(\hat{y}^*) + \hat{y}^*\underline{E}(\hat{y}^*)$$

The monopolist's optimal cutoff satisfies the FOC and hence

$$\bar{y} - \underline{E}(\hat{y}^*) = \hat{y}^* \frac{\partial \underline{E}(\hat{y}^*)}{\partial \hat{y}} \tag{1}$$

The derivative of $W_\nu(\hat{y}^*, z)$ with respect to z is:

$$\frac{dW_\nu(\hat{y}^*, z)}{dz} = (\nu - \hat{y}^*)\left(\frac{\partial \bar{E}}{\partial z}\right) + \hat{y}^* \frac{\partial \underline{E}}{\partial z} + \left[(\nu - \hat{y}^*)\frac{\partial \bar{E}}{\partial \hat{y}} - (\bar{E} - \underline{E}) + \hat{y}^* \frac{\partial \underline{E}}{\partial \hat{y}}\right] \frac{d\hat{y}^*}{dz}$$

Using (1) this becomes

$$\frac{dW_\nu(\hat{y}^*, z)}{dz} = (\nu - \hat{y}^*)\left(\frac{\partial \bar{E}}{\partial z}\right) + \hat{y}^* \frac{\partial \underline{E}}{\partial z} + \left[(\nu - \hat{y}^*)\frac{\partial \bar{E}}{\partial \hat{y}} - (\bar{E} - \bar{y})\right] \frac{d\hat{y}^*}{dz}$$

Hence, sufficient conditions for $W_\nu(\hat{y}^*, z)$ to be decreasing in z are that

$$(\nu - \hat{y}^*)\left(\frac{\partial \bar{E}}{\partial z}\right) + \hat{y}^* \frac{\partial \underline{E}}{\partial z} < 0$$

and

$$\left[(\nu - \hat{y}^*)\frac{\partial \bar{E}}{\partial \hat{y}} - (\bar{E} - \bar{y})\right] < 0$$

(because we already know that $\frac{d\hat{y}^*}{dz} > 0$). Again setting $\nu = 1$ and using

$$\frac{\partial \bar{E}}{\partial z} = \frac{\frac{1}{2}(1 - \hat{y})^2}{(z + (1 - 2z)(1 - \hat{y}))^2}$$

$$\frac{\partial \underline{E}}{\partial z} = \frac{-\hat{y}^2}{2(z + (1 - 2z)\hat{y})^2}$$

and

$$\frac{\partial \bar{E}}{\partial \hat{y}} = (1-2z)(1-\hat{y})\frac{z+(1-2z)\left(\frac{1}{2}-\frac{\hat{y}}{2}\right)}{(z+(1-2z)(1-\hat{y}))^2},$$

it can be shown that both terms are negative for $\hat{y}^* \in [0.5,1]$ and all $z \in [0,0.5]$: It turns out that

$$(\nu-\hat{y}^*)\left(\frac{\partial \bar{E}}{\partial z}\right)+\hat{y}^*\frac{\partial \underline{E}}{\partial z} = \frac{(1-\hat{y})^3}{2(z+(1-2z)(1-\hat{y}))^2} - \frac{\hat{y}^3}{2(z+(1-2z)\hat{y})^2}$$

$$= \frac{(1-\hat{y})^3(z+(1-2z)\hat{y})^2 - \hat{y}^3(z+(1-2z)(1-\hat{y}))^2}{2(z+(1-2z)(1-\hat{y}))^2(z+(1-2z)\hat{y})^2}.$$

The denominator is positive, and the numerator can be simplified to give

$$z^2(1-3\hat{y}+3\hat{y}^2-2\hat{y}^3)+(2z-4z^2)(\hat{y}-3\hat{y}^2+2\hat{y}^3)+(1-2z)^2\hat{y}^2(1-4\hat{y}+5\hat{y}^2-2\hat{y}^3).$$

It turns out that the polynomials of $\hat{y}$ in each summand are negative for all $\hat{y} \in [0.5,1]$, hence the expression is negative for the relevant ranges of $\hat{y}$ and all z.

Furthermore,

$$\left[(\nu-\hat{y}^*)\frac{\partial \bar{E}}{\partial \hat{y}} - (\bar{E}-\bar{y})\right] = (1-2z)(1-\hat{y})^2\frac{z+(1-2z)\left(\frac{1}{2}-\frac{\hat{y}}{2}\right)}{(z+(1-2z)(1-\hat{y}))^2}$$

$$-\frac{z+(1-2z)\left(\frac{1}{2}-\frac{\hat{y}^2}{2}\right)}{z+(1-2z)(1-\hat{y})}+\frac{1}{2}$$

$$= \frac{-z^2+(1-2z)z(1-4\hat{y}+3\hat{y}^2)+(1-2z)^2(1-4\hat{y}+5\hat{y}^2-2\hat{y}^3)}{2(z+(1-2z)(1-\hat{y}))^2}$$

Again the denominator is positive and all the polynomials of $\hat{y}$ in the numerator are negative $\forall \hat{y} \in [0.5,1]$, which implies that the expression is negative for the relevant ranges of $\hat{y}$ and all z. Hence, $\frac{dW_\nu(\hat{y}^*,z)}{dz}$ is negative for all $\hat{y}$ in $[0.5,1]$ for all z and thus $W_\nu(\hat{y}^*,z)$ is decreasing in z. ■

Finally, in addition to looking at the richest in society, how does an increase in inequality affect welfare of sorting at the monopolist's optimal cutoff for the poor group? We know that people in the poor group would always be better off without sorting, but how does average utility in the poor group change with inequality? Here, I find the following:

Proposition A7 *Average welfare in the poor group from sorting at the monopolist's optimal cutoff is decreasing in inequality.*

Proof. Average welfare in the poor group amounts to $\underline{E}^2$ (note that they don't have to pay the sorting fee b). We know that

$$\frac{d\underline{E}(\hat{y}^*, z)}{dz} = \frac{\partial \underline{E}}{\partial \hat{y}} \frac{d\hat{y}^*}{dz} + \frac{\partial \underline{E}}{\partial z}$$

From above we know that

$$\frac{\partial \underline{E}}{\partial \hat{y}} = \frac{(1-2z)\left(\hat{y}z + (1-2z)\frac{\hat{y}^2}{2}\right)}{\left(z + (1-2z)\hat{y}\right)^2}$$

and

$$\frac{\partial \underline{E}}{\partial z} = \frac{-\hat{y}^2}{2(z + (1-2z)\hat{y})^2}$$

and plugging in all the expressions for the derivatives in $\frac{d\hat{y}^*}{dz}$ yields

$$\frac{d\hat{y}^*}{dz} = \frac{\left(\frac{3z\hat{y}}{2} + (1-2z)\frac{\hat{y}^2}{2}\right)}{(1-2z)(3z^2 + 3(1-2z)\hat{y}z + (1-2z)^2\hat{y}^2)}$$

Hence, after simplifications, we get that

$$\frac{d\underline{E}(\hat{y}^*, z)}{dz} = \frac{-(1-2z)^2\frac{\hat{y}^4}{4} - (1-2z)\,\frac{\hat{y}^3 z}{4}}{(3z^2 + 3(1-2z)\hat{y}z + (1-2z)^2\hat{y}^2)\left(z + (1-2z)\hat{y}\right)^2}$$

The denominator is always positive and the numerator is always negative, hence $\frac{d\underline{E}(\hat{y}^*,z)}{dz} < 0$. ■

An increase in inequality has two effects on average welfare in the poor group: We know that the monopolist's optimal cutoff increases due to a rise in inequality, which benefits the poor group because people with higher incomes become members of their group and push average income up.[2] However, this increase in the cutoff is not enough to counteract the negative effect of an increasing mass of poor people with zero income in their group, which pulls average income and average welfare down. The overall effect of an increase in inequality is thus negative.

[2]Incidentally, it does of course *not* benefit those people who now become members of the poor group.

To conclude, the above calculations show that in case of the symmetric atoms distribution, an increase in inequality leads to a decline in total welfare from sorting, average welfare in the poor group and welfare of the richest individuals (if the monopolist chooses the cutoff to maximize her profits). For this (stylized) income distribution, an increase in inequality is thus never beneficial for welfare in the presence of sorting (at the monopolist's optimal cutoff).

2 Additional calculations for Section 4

2.1 Triangle distribution

If the density is

$$\begin{aligned} f(y) &= \frac{4}{\nu^2} y \quad \text{if } y \in \left[0, \frac{\nu}{2}\right] \\ f(y) &= \frac{4}{\nu} - \frac{4}{\nu^2} y \quad \text{if } y \in \left[\frac{\nu}{2}, \nu\right] \end{aligned}$$

then if the cutoff $\hat{y}$ is in $\left[0, \frac{\nu}{2}\right]$ the conditional expectations are

$$\underline{E}(\hat{y}) = \frac{2\hat{y}}{3}$$

and

$$\bar{E}(\hat{y}) = \frac{\frac{\nu}{2} - \frac{4\hat{y}^3}{3\nu^2}}{1 - \frac{2\hat{y}^2}{\nu^2}}$$

whereas if $\hat{y} \in \left[\frac{\nu}{2}, \nu\right]$ the expressions become

$$\underline{E}(\hat{y}) = \frac{\frac{2\hat{y}^2}{\nu} - \frac{4}{3}\frac{\hat{y}^3}{\nu^2} - \frac{\nu}{6}}{\frac{4\hat{y}}{\nu}\left(1 - \frac{\hat{y}}{2\nu}\right) - 1}$$

and

$$\bar{E}(\hat{y}) = \frac{\frac{2\nu}{3} - \frac{2\hat{y}^2}{\nu} + \frac{4}{3}\frac{\hat{y}^3}{\nu^2}}{2 - \frac{4\hat{y}}{\nu} + \frac{2\hat{y}^2}{\nu^2}}$$

Monopolist profits are

$$\Pi(\hat{y}) = \hat{y}(\bar{y} - \underline{E}(\hat{y})).$$

It is straightforward to show that $\Pi(.)$ reaches a local maximum at $\frac{3\nu}{8}$ if $\hat{y} \leq \frac{\nu}{2}$ and is decreasing in $\hat{y}$ for all $\hat{y} > \frac{\nu}{2}$. Hence, cutoff $\hat{y}^* = \frac{3\nu}{8}$ yields the maximal profit, and $\Pi(\hat{y}^*) = \frac{3}{32}\nu^2$. Welfare at this cutoff is given by

$$U^S(\hat{y}^*) = F(\hat{y}^*) \left(\underline{E}(\hat{y}^*)\right)^2 + (1-F(\hat{y}^*)) \left(\bar{E}(\hat{y}^*)\right)^2 - \Pi(\hat{y}^*) = \frac{\nu^2}{32}\left(\frac{3059}{529}\right) \approx 0.1807\nu^2$$

2.2 Uniform distribution

We have that

$$\underline{E}(\hat{y}) = \frac{\hat{y}}{2}$$

and

$$\bar{E}(\hat{y}) = \frac{\nu + \hat{y}}{2}$$

and thus

$$\Pi(\hat{y}) = \frac{\nu\hat{y}}{2} - \frac{\hat{y}^2}{2}$$

which is maximized at

$$\hat{y}^* = \frac{\nu}{2}.$$

Welfare at $\hat{y}^*$ is

$$U^S(\hat{y}^*) = \frac{3\nu^2}{16}.$$

2.3 Reverse triangle distribution

If the density is

$$\begin{aligned} f(y) &= \frac{2}{\nu} - \frac{4}{\nu^2}y \qquad \text{if } y \in \left[0, \frac{\nu}{2}\right] \\ f(y) &= -\frac{2}{\nu} + \frac{4}{\nu^2}y \qquad \text{if } y \in \left[\frac{\nu}{2}, \nu\right] \end{aligned}$$

then if the cutoff $\hat{y}$ is in $\left[0, \frac{\nu}{2}\right]$ the conditional expectations are

$$\underline{E}(\hat{y}) = \frac{\frac{\hat{y}}{2} - \frac{2}{3}\frac{\hat{y}^2}{\nu}}{1 - \frac{\hat{y}}{\nu}}$$

and

$$\bar{E}(\hat{y}) = \frac{\frac{\nu}{2} - \frac{2\hat{y}}{\nu}\left(\frac{\hat{y}}{2} - \frac{2}{3}\frac{\hat{y}^2}{\nu}\right)}{1 - \frac{2\hat{y}}{\nu}\left(1 - \frac{\hat{y}}{\nu}\right)}$$

whereas if $\hat{y} \in \left[\frac{\nu}{2}, \nu\right]$ the expressions become

$$\underline{E}(\hat{y}) = \frac{\frac{\nu}{6} - \frac{2\hat{y}^2}{\nu}\left(\frac{1}{2} - \frac{2}{3}\frac{\hat{y}}{\nu}\right)}{1 - \frac{2\hat{y}}{\nu}\left(1 - \frac{\hat{y}}{\nu}\right)}$$

and

$$\bar{E}(\hat{y}) = \frac{\frac{\nu}{3} + \frac{2\hat{y}^2}{\nu}\left(\frac{1}{2} - \frac{2}{3}\frac{\hat{y}}{\nu}\right)}{\frac{2\hat{y}}{\nu}\left(1 - \frac{\hat{y}}{\nu}\right)}.$$

Monopolist profits are

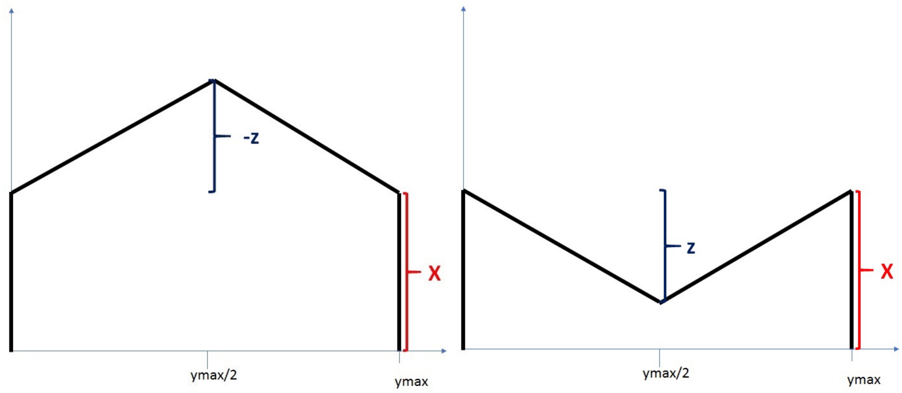
$$\Pi(\hat{y}) = \hat{y}(\bar{y} - \underline{E}(\hat{y})).$$

It is straightforward to show that $\Pi(.)$ reaches a local maximum at 0.64ν (numerically calculated) if $\hat{y} > \frac{\nu}{2}$ and is decreasing in $\hat{y}$ for all $\hat{y} \leq \frac{\nu}{2}$. Hence, cutoff $\hat{y}^* = 0.64\nu$ yields the maximal profit, and $\Pi(\hat{y}^*) \approx 0.1935\nu^2$. Welfare at this cutoff is given by

$$U^S(\hat{y}^*) = F(\hat{y}^*)\left(\underline{E}(\hat{y}^*)\right)^2 + (1 - F(\hat{y}^*))\left(\bar{E}(\hat{y}^*)\right)^2 - \Pi(\hat{y}^*) \approx 0.163\nu^2$$

2.4 Continuously increasing inequality

The following section examines an income distribution in which inequality can be varied continuously and which encompasses the triangle, uniform and reverse triangle distribution discussed in Section 4. It demonstrates that the findings from Section 4 concerning discrete jumps in inequality translate also to this continous version: The monopolist's profit and the optimal cutoff are increasing in inequality. Total welfare (from sorting at the monopolist's optimal cutoff) is increasing in inequality for low rates of inequality, but decreases for high rates. In addition, I find that, similar to total welfare, welfare from sorting for the richest individuals increases with inequality for low rates and decreases for high rates. On the other hand, average welfare in the poor group decreases with inequality.



(a) House distribution (b) Trough distribution

Figure 1: Sketch of the house and the trough distribution

2.4.1 Profit-maximizing cutoff

Suppose income is distributed on $[0, \nu]$ according to an income distribution with pdf $f(.)$ such that

$$\begin{aligned} f(y) &= x - \frac{2z}{\nu} y \qquad \text{if } y \in \left[0, \frac{\nu}{2}\right] \\ f(y) &= x - 2z + \frac{2z}{\nu} y \qquad \text{if } y \in \left[\frac{\nu}{2}, \nu\right] \end{aligned}$$

Note that we must have

$$x = \frac{1}{\nu} + \frac{z}{2} \tag{2}$$

in order for $F(\nu) = 1$ and

$$z \in \left[-\frac{2}{\nu}, \frac{2}{\nu}\right]$$

which implies $x \in \left[0, \frac{2}{\nu}\right]$. If $z = 0$ then $x = \frac{1}{\nu}$ and the distribution is uniform, if $z = -\frac{2}{\nu}$ then x is 0 and the pdf has the shape of an isosceles triangle. If $z = \frac{2}{\nu}$ then $x = \frac{2}{\nu}$ and the pdf has the shape of an inverse triangle. If $z \in \left(-\frac{2}{\nu}, 0\right)$ then the distribution has the shape of a house, if $z \in \left(0, \frac{2}{\nu}\right)$ the distribution has the shape of a (triangular) trough (see Figures 1a and 1b). The larger z, the higher is inequality (in terms of $\bar{E} - \underline{E}$ for any given cutoff) and an increase in z amounts to a (monotone) mean-preserving spread of the income distribution. Note that average income is constant $\forall z \in \left[-\frac{2}{\nu}, \frac{2}{\nu}\right]$, $\bar{y} = \frac{\nu}{2}$.

Using (2), the pdf can be rewritten as

$$\begin{aligned} f(y) &= \frac{1}{\nu}+\frac{z}{2}-\frac{2z}{\nu}y \qquad \text{if } y\in\left[0,\frac{\nu}{2}\right] \\ f(y) &= \frac{1}{\nu}-\frac{3z}{2}+\frac{2z}{\nu}y \quad \text{if } y\in\left[\frac{\nu}{2},\nu\right] \end{aligned}$$

If the cutoff $\hat{y}$ is in the interval $\left[0,\frac{\nu}{2}\right]$, we have that

$$\underline{E}(\hat{y})=\frac{\int_0^{\hat{y}}\left(\frac{1}{\nu}+\frac{z}{2}-\frac{2z}{\nu}y\right)ydy}{F(\hat{y})}=\frac{\int_0^{\hat{y}}\left(\frac{1}{\nu}+\frac{z}{2}-\frac{2z}{\nu}y\right)ydy}{\int_0^{\hat{y}}\left(\frac{1}{\nu}+\frac{z}{2}-\frac{2z}{\nu}y\right)dy}$$

$$=\frac{\frac{\hat{y}^2}{2\nu}+\frac{\hat{y}^2z}{4}-\frac{2z\hat{y}^3}{3\nu}}{\frac{\hat{y}(1-z\hat{y})}{\nu}+\frac{\hat{y}z}{2}}=\frac{6\hat{y}+3\hat{y}z\nu-8z\hat{y}^2}{12-12z\hat{y}+6\nu z}$$

If the cutoff is above $\frac{\nu}{2}$ we need to calculate $\underline{E}(\hat{y})$ differently: The easiest way is to calculate $\bar{E}\left(\hat{y}\right)$ first

$$\bar{E}(\hat{y})=\frac{\int_{\hat{y}}^{\nu}\left(\frac{1}{\nu}-\frac{3z}{2}+\frac{2z}{\nu}y\right)ydy}{1-F(\hat{y})}=\frac{\int_{\hat{y}}^{\nu}\left(\frac{1}{\nu}-\frac{3z}{2}+\frac{2z}{\nu}y\right)ydy}{\int_{\hat{y}}^{\nu}\left(\frac{1}{\nu}-\frac{3z}{2}+\frac{2z}{\nu}y\right)dy}$$

$$=\frac{\frac{\nu}{2}-\frac{1}{12}z\nu^2-\frac{\hat{y}^2}{2\nu}+\frac{3}{4}z\hat{y}^2-\frac{2}{3}\frac{z}{\nu}\hat{y}^3}{1-\frac{1}{2}z\nu-\frac{\hat{y}}{\nu}+\frac{3z\hat{y}}{2}-\frac{z\hat{y}^2}{\nu}}$$

and then calculate $\underline{E}(\hat{y})$ via the formula

$$\bar{y}=F\left(\hat{y}\right)\underline{E}(\hat{y})+(1-F(\hat{y}))\bar{E}(\hat{y})$$

(noting that $\bar{y}=\frac{\nu}{2}$), which gives

$$\underline{E}(\hat{y})=\frac{\frac{1}{12}z\nu^2+\frac{\hat{y}^2}{2\nu}-\frac{3}{4}z\hat{y}^2+\frac{2}{3}\frac{z}{\nu}\hat{y}^3}{\frac{1}{2}z\nu+\frac{\hat{y}}{\nu}-\frac{3z\hat{y}}{2}+\frac{z\hat{y}^2}{\nu}}$$

$$=\frac{z\nu^2+\frac{6\hat{y}^2}{\nu}-9z\hat{y}^2+\frac{8z\hat{y}^3}{\nu}}{6z\nu+\frac{12\hat{y}}{\nu}-18z\hat{y}+\frac{12\hat{y}^2z}{\nu}}.$$

Using these expressions, we can show the following:

Proposition A8 *If $z \in \left[-\frac{2}{\nu}, 0\right]$, the monopolist's optimal cutoff is in the interval $\left[0, \frac{\nu}{2}\right]$, if $z = 0$ the monopolist's optimal cutoff is $\hat{y}^* = \frac{\nu}{2}$ and if $z \in \left[0, \frac{2}{\nu}\right]$, the monopolist's optimal cutoff is in the interval $\left[\frac{\nu}{2}, \nu\right]$.*

Proof. The monopolist's profit at cutoff $\hat{y}$ is given by

$$\Pi(\hat{y}) = \hat{y}(\bar{y} - \underline{E})$$

Using the expressions for $\underline{E}$ from above, we find that

$$\Pi(\hat{y}) = \hat{y}\left(\frac{\nu}{2} - \frac{6\hat{y} + 3\hat{y}z\nu - 8z\hat{y}^2}{12 - 12z\hat{y} + 6\nu z}\right) \tag{3}$$

if $\hat{y} \in \left[0, \frac{\nu}{2}\right]$ and

$$\Pi(\hat{y}) = \hat{y}\left(\frac{\nu}{2} - \frac{z\nu^2 + \frac{6\hat{y}^2}{\nu} - 9z\hat{y}^2 + \frac{8z\hat{y}^3}{\nu}}{6z\nu + \frac{12\hat{y}}{\nu} - 18z\hat{y} + \frac{12\hat{y}^2 z}{\nu}}\right) \tag{4}$$

if $\hat{y} \in \left[\frac{\nu}{2}, \nu\right]$. It can be calculated (numerically) that (3) has a local and global maximum in $\left[0, \frac{\nu}{2}\right]$ when $z < 0$, while (4) has a local and global maximum in $\left[\frac{\nu}{2}, \nu\right]$ when $z > 0$. ■

Proposition A9 *The monopolist's profit-maximizing cutoff $\hat{y}^*$ is increasing in z for all $z \in \left[-\frac{2}{\nu}, \frac{2}{\nu}\right]$.*

Proof. Note that

$$\frac{d\hat{y}^*}{dz} = \frac{-\frac{\partial \underline{E}(\hat{y}^*, z)}{\partial z} - \hat{y}^* \frac{\partial^2 \underline{E}(\hat{y}^*, z)}{\partial \hat{y} \partial z}}{\hat{y}^* \frac{\partial^2 \underline{E}(\hat{y}, z)}{(\partial \hat{y})^2} + 2\frac{\partial \underline{E}(\hat{y}^*, z)}{\partial \hat{y}}}$$

and hence according to Lemma A1, sufficient conditions for

$$\frac{d\hat{y}^*}{dz} > 0$$

are

$$\frac{\partial^2 \underline{E}(\hat{y}, z)}{\partial \hat{y} \partial z} \leq 0 \quad \text{and} \quad \frac{\partial^2 \underline{E}(\hat{y}, z)}{(\partial \hat{y})^2} \geq 0.$$

Show $\frac{\partial^2 \underline{E}(\hat{y}, z)}{(\partial \hat{y})^2} \geq 0$ if $z < 0$:

If $z < 0$ we know that the monopolist's optimal cutoff is in the interval

$[0, \frac{\nu}{2})$. Setting $\nu = 1$ again for simplicity of notation we have that

$$\underline{E}(\hat{y}, z) = \frac{6\hat{y} + 3\hat{y}z - 8z\hat{y}^2}{12 - 12z\hat{y} + 6z}$$

$$\begin{aligned} \frac{\partial \underline{E}(\hat{y}, z)}{\partial \hat{y}} &= \frac{(6 + 3z - 16z\hat{y})(12 - 12z\hat{y} + 6z) + (6\hat{y} + 3\hat{y}z - 8z\hat{y}^2)12z}{(12 - 12z\hat{y} + 6z)^2} \\ &= 6\frac{12 + 12z - 32z\hat{y} - 16z^2\hat{y} + 16z^2\hat{y}^2 + 3z^2}{(12 - 12z\hat{y} + 6z)^2} \end{aligned}$$

Therefore

$$\frac{\partial^2 \underline{E}(\hat{y}, z)}{(\partial \hat{y})^2} = 6.\frac{\left[\begin{array}{c} (-32z - 16z^2 + 32z^2\hat{y})(12 - 12z\hat{y} + 6z) \\ +24z(12 + 12z - 32z\hat{y} - 16z^2\hat{y} + 16z^2\hat{y}^2 + 3z^2) \end{array} \right]}{(12 - 12z\hat{y} + 6z)^3}$$

It is immediate to see that

$$12 - 12z\hat{y} + 6z > 0$$

for all $z < 0$, i.e. $z \in [-2, 0]$, therefore it suffices to examine the numerator of this expression. The numerator can be rewritten as

$$36(-16z - 16z^2 - 4z^3)$$

which is always positive if $z < 0$. Thus, $\frac{\partial^2 \underline{E}(\hat{y}, z)}{(\partial \hat{y})^2} > 0$ if $z < 0$.

Show $\frac{\partial^2 \underline{E}(\hat{y}, z)}{\partial \hat{y} \partial z} \leq 0$ if $z < 0$:

Given that

$$\frac{\partial \underline{E}(\hat{y}, z)}{\partial \hat{y}} = 6\frac{12 + 12z - 32z\hat{y} - 16z^2\hat{y} + 16z^2\hat{y}^2 + 3z^2}{(12 - 12z\hat{y} + 6z)^2}$$

I can calculate

$$\frac{\partial^2 \underline{E}(\hat{y}, z)}{\partial \hat{y} \partial z} = 6\frac{\left[\begin{array}{c} (12 - 32\hat{y} - 32z\hat{y} + 32z\hat{y}^2 + 6z)(12 - 12z\hat{y} + 6z) \\ -2(-12\hat{y} + 6)(12 + 12z - 32z\hat{y} - 16z^2\hat{y} + 16z^2\hat{y}^2 + 3z^2) \end{array} \right]}{(12 - 12z\hat{y} + 6z)^3}$$

Again it suffices to examine the numerator, which can be rewritten as

$$36[(12 - 32\hat{y} - 32z\hat{y} + 32z\hat{y}^2 + 6z)(2 - 2z\hat{y} + z)$$
$$-(-4\hat{y} + 2)(12 + 12z - 32z\hat{y} - 16z^2\hat{y} + 16z^2\hat{y}^2 + 3z^2)]$$

and simplified to

$$36\hat{y}(-16 - 8z)$$

which is always negative if $z \in [-2, 0]$. Thus, $\frac{\partial^2 \underline{E}(\hat{y},z)}{\partial \hat{y} \partial z} < 0$ if $z < 0$.

As both sufficient conditions hold, we have that $\frac{d\hat{y}^*}{dz} > 0$ if $z < 0$.

Show $\frac{\partial^2 \underline{E}(\hat{y},z)}{(\partial \hat{y})^2} \geq 0$ if $z > 0$:

If $z > 0$ we know the monopolist's optimal cutoff lies above $\frac{\nu}{2}$ and therefore (again setting $\nu = 1$)

$$\underline{E}(\hat{y}, z) = \frac{z + 6\hat{y}^2 - 9z\hat{y}^2 + 8z\hat{y}^3}{6z + 12\hat{y} - 18z\hat{y} + 12\hat{y}^2 z}$$

Therefore we have

$$\frac{\partial \underline{E}(\hat{y}, z)}{\partial \hat{y}} = \frac{6}{(6z + 12\hat{y} - 18z\hat{y} + 12\hat{y}^2 z)^2}[-2z + 3z^2 + 12\hat{y}^2 + 12zy - 22z^2\hat{y} + 51z^2\hat{y}^2 -36z\hat{y}^2 - 48z^2\hat{y}^3 + 16z^2\hat{y}^4 + 32z\hat{y}^3]$$

and

$$\frac{\partial^2 \underline{E}(\hat{y}, z)}{(\partial \hat{y})^2} = \frac{36}{(6z + 12\hat{y} - 18z\hat{y} + 12\hat{y}^2 z)^3} \times [(24\hat{y} + 12z - 22z^2 + 102z^2\hat{y} - 72z\hat{y} - 144z^2\hat{y}^2 + 64z^2\hat{y}^3 + 96z\hat{y}^2)(z + 2\hat{y} - 3z\hat{y} + 2z\hat{y}^2) +(-4 + 6z - 8z\hat{y})(-2z + 3z^2 + 12\hat{y}^2 + 12zy - 22z^2\hat{y} +51z^2\hat{y}^2 - 36z\hat{y}^2 - 48z^2\hat{y}^3 + 16z^2\hat{y}^4 + 32z\hat{y}^3)]$$

which can be shown to be positive $\forall y \in [0.5, 1]$ and $\forall z > 0$.

Show $\frac{\partial^2 \underline{E}(\hat{y},z)}{\partial \hat{y} \partial z} \leq 0$ if $z > 0$:

$$\frac{\partial \underline{E}(\hat{y}, z)}{\partial z} = \frac{6(2\hat{y} + 4\hat{y}^4 - 6\hat{y}^2)}{(6z + 12\hat{y} - 18z\hat{y} + 12\hat{y}^2 z)^2}$$

and hence

$$\frac{\partial^2 \underline{E}(\hat{y},z)}{\partial z \partial y} = \frac{36}{(6z+12\hat{y}-18z\hat{y}+12\hat{y}^2 z)^3}[(2+16\hat{y}^3-12\hat{y})(z+2\hat{y}-3z\hat{y}+2z\hat{y}^2) -2(2\hat{y}+4\hat{y}^4-6\hat{y}^2)(2-3z+4z\hat{y})]$$

Unfortunately, this is not always negative. In fact for high $\hat{y}$ it can be seen from simulations that it is positive for all z. The intuition for this is that the shape of the distribution is that of a trough in this case, and as z increases the trough becomes deeper. This means that there is a lot of mass higher up in the income distribution, and as the cutoff moves towards there, average income in the poor group increases due to this. This means that one of the sufficient conditions doesn't hold in the case of $z > 0$, so we need to calculate the whole expression for $\frac{d\hat{y}}{dz}$ to prove that it is positive. Plugging all the derivatives into this expression yields indeed that $\frac{d\hat{y}^*}{dz} > 0$ for all z (numerically calculated - note that the maximum $\hat{y}^*$ is at 0.6427051, when $z = -2$). ■

2.4.2 Welfare

We already know from Proposition 2 that an increase in inequality (resp. z) increases the monopolist's maximized profits. But what happens to total welfare, welfare of the richest and average welfare in the poor group? The following propositions show that total welfare from sorting at the monopolist's optimal cutoff increases with inequality for small rates of inequality and decreases if inequality is high. This development is driven by welfare in the rich group (and in particular of the richest individuals), which moves in the same way. Average welfare in the poor group decreases with inequality for all z.

Proposition A10 *Total welfare from sorting at the monopolist's optimal cutoff is increasing in z if $z \in \left[-\frac{2}{\nu}, 0\right]$.*

Proof. The derivative of total welfare with respect to z at the monopolist's optimal cutoff $\hat{y}^*$ amounts to

$$\frac{dTW(\hat{y}^*,z)}{dz} = \left(f(\underline{E}^2-\bar{E}^2)+F2\underline{E}\frac{\partial \underline{E}}{\partial \hat{y}^*}+(1-F)2\bar{E}\frac{\partial \bar{E}}{\partial \hat{y}^*}\right)\frac{d\hat{y}^*}{dz}$$

$$+\frac{\partial F}{\partial z}(\underline{E}^2-\bar{E}^2)+F2\underline{E}\frac{\partial \underline{E}}{\partial z}+(1-F)2\bar{E}\frac{\partial \bar{E}}{\partial z}-\frac{d\Pi(\hat{y}^*,z)}{dz}$$

where $\Pi(\hat{y}^*,z)$ is the monopolist's maximized profit and we know that

$$\frac{d\Pi(\hat{y}^*,z)}{dz}=-\hat{y}^*\frac{\partial \underline{E}}{\partial z}>0$$

Hence, the above expression can be simplified to

$$\frac{dTW(\hat{y}^*,z)}{dz}=f(\bar{E}-\underline{E})(\bar{E}+\underline{E}-2\hat{y}^*)\frac{d\hat{y}^*}{dz}+$$

$$+\frac{\partial F}{\partial z}(\underline{E}^2-\bar{E}^2)+(F2\underline{E}+\hat{y}^*)\frac{\partial \underline{E}}{\partial z}+(1-F)2\bar{E}\frac{\partial \bar{E}}{\partial z}$$

Note that if $z<0$ we know that $\hat{y}^*<\frac{\nu}{2}$.

(Set $\nu=1$ again) We have

$$\bar{E}=\frac{6-6\hat{y}^2-3z\hat{y}^2+8z\hat{y}^3}{12-12\hat{y}-6z\hat{y}+12z\hat{y}^2}$$

and

$$\frac{\partial \bar{E}}{\partial z}=\frac{6}{(12-12\hat{y}-6z\hat{y}+12z\hat{y}^2)^2}\hat{y}(6-18\hat{y}-4\hat{y}^3+16\hat{y}^2)$$

Furthermore

$$\frac{\partial \underline{E}}{\partial z}=\frac{6}{(12-6z+12z\hat{y})^2}\hat{y}(-4\hat{y}^2)$$

and note that

$$F(\hat{y},z)=\hat{y}+\frac{z\hat{y}}{2}-z\hat{y}^2$$

and hence

$$\frac{\partial F}{\partial z}=\frac{\hat{y}}{2}-\hat{y}^2>0 \quad \forall \hat{y}\in[0,0.5]$$

Note that $\frac{\partial \underline{E}}{\partial z}<0$ but $\frac{\partial F}{\partial z}>0$ because $\hat{y}^*<\frac{\nu}{2}$. As I have shown above that $\frac{d\hat{y}^*}{dz}>0\ \forall z$, sufficient conditions for $\frac{dTW(\hat{y}^*,z)}{dz}>0$ are that

$$\bar{E}+\underline{E}-2\hat{y}^*>0 \tag{5}$$

and

$$\frac{\partial F}{\partial z}(\underline{E}^2-\bar{E}^2)+(F2\underline{E}+\hat{y}^*)\frac{\partial \underline{E}}{\partial z}+(1-F)2\bar{E}\frac{\partial \bar{E}}{\partial z}>0 \tag{6}$$

It is easy to check that condition (5) always holds in this case. After some algebra, it can be seen from numerical calculations that also (6) holds. Hence, $\frac{dTW(\hat{y}^*,z)}{dz} > 0$ if $z < 0$. ■

Proposition A11 *Total welfare from sorting at the monopolist's optimal cutoff is decreasing in z if* $z \in \left[0, \frac{2}{\nu}\right]$.

Proof. Note that if $z > 0$ we know that $\hat{y} > \frac{\nu}{2}$. As above we have

$$\frac{dTW(\hat{y}^*,z)}{dz} = f(\bar{E} - \underline{E})(\bar{E} + \underline{E} - 2\hat{y}^*)\frac{d\hat{y}^*}{dz} +$$

$$+\frac{\partial F}{\partial z}(\underline{E}^2 - \bar{E}^2) + (F2\underline{E} + \hat{y}^*)\frac{\partial \underline{E}}{\partial z} + (1-F)2\bar{E}\frac{\partial \bar{E}}{\partial z}$$

Sufficient conditions for $\frac{dTW(\hat{y}^*,z)}{dz} < 0$ are that

$$\bar{E} + \underline{E} - 2\hat{y}^* < 0 \tag{7}$$

and

$$\frac{\partial F}{\partial z}(\underline{E}^2 - \bar{E}^2) + (F2\underline{E} + \hat{y}^*)\frac{\partial \underline{E}}{\partial z} + (1-F)2\bar{E}\frac{\partial \bar{E}}{\partial z} < 0. \tag{8}$$

Note that in this case we have (again setting $\nu = 1$) that

$$\bar{E} = \frac{6 - z - 6\hat{y}^2 + 9z\hat{y}^2 - 8z\hat{y}^3}{12 - 6z - 12\hat{y} + 18z\hat{y} - 12z\hat{y}^2}$$

and

$$\underline{E} = \frac{z + 6\hat{y}^2 - 9z\hat{y}^2 + 8z\hat{y}^3}{6z + 12\hat{y} - 18z\hat{y} + 12\hat{y}^2 z}.$$

Hence

$$\frac{\partial \underline{E}}{\partial z} = \frac{6}{(6z + 12\hat{y} - 18z\hat{y} + 12\hat{y}^2 z)^2}(2\hat{y} + 4\hat{y}^4 - 6\hat{y}^2)$$

and

$$\frac{\partial \bar{E}}{\partial z} = \frac{6}{(12 - 6z - 12\hat{y} + 18z\hat{y} - 12z\hat{y}^2)^2}(4 - 16\hat{y} - 16\hat{y}^3 + 24\hat{y}^2 + 16\hat{y}^4).$$

Furthermore, note that

$$F(\hat{y}, z) = \frac{1}{2}z + \hat{y} - \frac{3z\hat{y}}{2} + z\hat{y}^2$$

and hence

$$\frac{\partial F}{\partial z} = \frac{1}{2} - \frac{3\hat{y}}{2} + \hat{y}^2.$$

Plugging in these expressions, it can easily be shown that (7) is always negative. However, concerning (8), there is a small range of $\hat{y} > 0.5$ for which this expression is positive. Indeed it can be shown (in numerical simulations) that the whole expression $\frac{dTW(\hat{y}^*,z)}{dz}$ is positive for all z for small $\hat{y}^* > 0.5$. However, note that the monopolist's optimal cutoff increases with z as well, and this increase moves $\hat{y}^*$ out of the area for which total welfare increases with z also for very small z. In fact, for all $z > 0$ it can be shown (again numerically) that $\hat{y}^*$ is greater than the small range of $\hat{y}$ for which $\frac{dTW(\hat{y}^*,z)}{dz}$ would be positive. Hence, total welfare from sorting at the monopolist's optimal cutoff decreases with z if $z > 0$. ■

Proposition A12 *Welfare of the richest individuals from sorting at the monopolist's optimum is increasing in z for low rates of inequality and decreasing in z for high rates of inequality.*

Proof.

$$W_\nu(\hat{y}^*, z) = (\nu - \hat{y}^*)\bar{E}(\hat{y}^*, z) + \hat{y}^* \underline{E}(\hat{y}^*, z)$$

The monopolist's optimal cutoff satisfies the FOC and hence

$$\bar{y} - \underline{E}(\hat{y}^*, z) = \hat{y}^* \frac{\partial \underline{E}(\hat{y}^*, z)}{\partial \hat{y}} \tag{9}$$

The derivative of $W_\nu(\hat{y}^*)$ with respect to z is:

$$\frac{dW_\nu(\hat{y}^*, z)}{dz} = (\nu - \hat{y}^*) \left(\frac{\partial \bar{E}}{\partial z} \right) + \hat{y}^* \frac{\partial \underline{E}}{\partial z} + \left[(\nu - \hat{y}^*) \frac{\partial \bar{E}}{\partial \hat{y}} - (\bar{E} - \underline{E}) + \hat{y}^* \frac{\partial \underline{E}}{\partial \hat{y}} \right] \frac{d\hat{y}^*}{dz}$$

Using (9) this becomes

$$\frac{dW_\nu(\hat{y}^*, z)}{dz} = (\nu - \hat{y}^*) \left(\frac{\partial \bar{E}}{\partial z} \right) + \hat{y}^* \frac{\partial \underline{E}}{\partial z} + \left[(\nu - \hat{y}^*) \frac{\partial \bar{E}}{\partial \hat{y}} - (\bar{E} - \bar{y}) \right] \frac{d\hat{y}^*}{dz}$$

Hence, sufficient conditions for $W_\nu(\hat{y}^*)$ to be decreasing in z are that

$$(\nu - \hat{y}^*) \left(\frac{\partial \bar{E}}{\partial z} \right) + \hat{y}^* \frac{\partial \underline{E}}{\partial z} < 0$$

and

$$\left[(\nu - \hat{y}^*)\frac{\partial \bar{E}}{\partial \hat{y}} - (\bar{E} - \bar{y})\right] < 0$$

(because we already know that $\frac{d\hat{y}^*}{dz} > 0$).

For $z > 0$ it can be shown (numerically) that both terms are negative for $\hat{y}^* \in [0.5, 1]$ and all $z \in [0, 0.5]$. Hence, $\frac{dW_\nu(\hat{y}^*, z)}{dz}$ is negative for all $\hat{y}$ in $[0.5, 1]$ for all z and thus $W_\nu(\hat{y}^*, z)$ is decreasing in z: As inequality increases, welfare of the richest in society from sorting at the monopolist's optimal cutoff goes down.

For $z < 0$ these sufficient conditions don't hold. In fact it can be shown (numerically) that except for very small $z < -1.9$, welfare of the richest in society from sorting at the monopolist's optimal cutoff increases due to an increase in inequality. ■

Proposition A12 helps to understand the effect of an increase in inequality on the rich in the presence of sorting: as inequality increases, the monopolist increases the cutoff due to an increase in inequality, because the amount by which she can raise the sorting fee is higher than her loss of "customers" (= members of the rich group, who pay the fee). The increase in the cutoff benefits the rich group, but the increase in the sorting fee harms them. For low rates of inequality, the former effect is higher than the latter, hence welfare of the rich increases with inequality, but if inequality becomes too high (which, because it is in the form of a mean-preserving spread, means that there are more rich people as well as more poor) membership of their exclusive group becomes too expensive and the second effect dominates, leading to a negative relationship between inequality and welfare of the rich.

Proposition A13 *Average welfare in the poor group from sorting at the monopolist's optimum decreases in inequality.*

Proof. Average welfare in the poor group amounts to $\underline{E}^2$ (note that they don't have to pay b). We know that

$$\frac{d\underline{E}(\hat{y}^*, z)}{dz} = \frac{\partial \underline{E}}{\partial \hat{y}^*}\frac{d\hat{y}^*}{dz} + \frac{\partial \underline{E}}{\partial z} \tag{10}$$

Plugging in the expressions derived above, it is straightforward to show that (10) is negative for all z and all $\hat{y}^* > 0.5$. The intuition for this result is that, even though an increase in $\hat{y}^*$ actually benefits the poor group (because they get to interact with richer people on average), this is not enough to counteract the negative effect of an increasing mass of poor people with zero income in their group. The overall effect of an increase in inequality is thus negative. ■

3 Additional reproduction of proofs

3.1 Proof of Proposition 5 (Hoppe et al. (2009)

Total welfare from perfect sorting is given by (see main paper Appendix A.2.1.)

$$\int_0^{\nu}(y^2 - \frac{y^2}{2})f(y)dy = \int_0^{\nu}\frac{y^2}{2}f(y)dy = \frac{E(y^2)}{2}$$

Welfare from no sorting is

$$\int (y\bar{y})f(y)dy = \bar{y}^2$$

Welfare from perfect sorting is thus higher than welfare from no sorting iff

$$\frac{E(y^2)}{2} \geq \bar{y}^2 \tag{11}$$

Using the fact that $Var(y) = E(y^2) - \bar{y}^2$ we get that equation 11 holds if and only if

$$\frac{Var(y)}{\bar{y}^2} \geq 1 \iff CV \geq 1$$

3.2 Proof of Proposition 7 (Levy and Razin (2015))

Levy and Razin (2015) show that the difference Δ between welfare of sorting at any partition with finitely many cutoffs $y_1, ...y_i, ..y_{n-1}$ and welfare of no sorting can be written as

$$\Delta = \sum_{i=1}^{n-1}(E_{i-1} - E_i)(1 - F(y_i))(\bar{y} + y_i - \bar{E}_i)$$

where $E_i = E[y|y_{i-1} \leq y \leq y_i]$ and $\bar{E}_i = E[y|y \geq y_i]$. This difference is negative (and hence no sorting is better for welfare than any finite sorting) iff $(\bar{y} + y_i - \bar{E}_i)$ is positive at any y_i, which is exaclty the NBUE property.

3.3 Summary of Rayo (2013)

The argument behind Rayo's proof (2013) is that if an interval of Y is pooled rather than separated (by perfect sorting), people with income higher than the average income in that interval receive a lower utility relative to perfect sorting, while the opposite holds for people with below average income in that interval. Hence, the monopolist can reap more money from people below average if she decides to pool an interval, while she gets less from those above average income. In addition, if she pools an interval she can reap more money from people with income above that interval, as they are willing to pay more not fall into that group with (relatively) low average income. These gains and losses are weighted by the function $h(.)$. If h is decreasing, the net effect of pooling an interval is positive for the monopolist, if h is increasing, the net effect is negative. Hence, a necessary and sufficient condition for perfect sorting being optimal for the monopolist is that h must be nondecreasing everywhere.

Note that if h is decreasing, pooling some intervals is always better than perfect sorting for the monopolist. However, the intervals for which it is optimal to pool don't have to align with the intervals for which h is decreasing. Rayo (2013) presents a way to calculate the intervals for which pooling is better than perfect sorting. He shows that an interval for which pooling is optimal always has to be followed by an interval of perfect sorting, i.e. two pooling intervals can never be directly adjacent. In addition, he shows that it is never optimal to pool at the top.