Sufficient Statistics for Nonlinear Tax Systems with Preference Heterogeneity

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Abstract

A prominent justification for taxation of capital income, savings, bequests, and certain commodities is that taxes on consumption categories preferred by those with higher earnings ability are efficient even in the presence of nonlinear earnings taxation. This paper provides a method for characterizing optimal nonlinear tax systems in the presence of correlated preference heterogeneity, using sufficient statistics that can be estimated from behavioral responses to tax reforms. Our results encompass tax systems that implement the optimal mechanism, as well as simpler tax systems such as those that involve a nonlinear earnings tax and a separable nonlinear capital income tax, or those that involve a nonlinear earnings tax and an earnings-dependent but otherwise linear capital income tax. All optimal tax systems can be expressed using a simple sufficient statistic for preference heterogeneity: the difference between the cross-sectional variation of consumption (or saving) with income, and the causal effect of income on consumption (or savings). Our formulas for optimal differential commodity taxes produce empirically-implementable generalizations of the Atkinson-Stiglitz theorem, and take a familiar form that resembles the formula for the optimal nonlinear earnings tax. We extend our results to incorporate multidimensional heterogeneity and government motives to encourage savings. In a calibrated model of the U.S. economy, our formulas suggest that savings tax rates in the U.S. should be approximately zero at low income levels, but should increase with earned income.

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1 Introduction

Most tax systems around the world involve taxes on capital income, estates, inheritances, and on some categories of consumption. The seminal theorem of Atkinson and Stiglitz (1976) suggests that this could be suboptimal: the theorem states that when preferences are homogeneous and weakly separable, an optimal tax system will involve an income tax only. However, as was understood in contemporaneous work by Mirrlees (1976),[1] and more recently explored by Saez (2002), Diamond and Spinnewijn (2011), Gauthier and Henriet (2018) and others, the Atkinson-Stiglitz Theorem does not generalize when tastes for certain commodities (including future consumption) are correlated with earnings ability. Yet while economists have made significant progress on empirically-implementable formulas for nonlinear earnings taxes (e.g., Saez, 2001), the Atkinson-Stiglitz Theorem remains perhaps the sharpest practically implementable result about optimal commodity taxation, as existing insights about the limitations of this theorem are mostly qualitative,[2] or rely on structural models with strong assumptions about the functional form of utility.

In this paper, we fill this gap by developing sufficient statistics formulas for optimal commodity taxation in the presence of nonlinear taxation. Our formulas nest the Atkinson-Stiglitz Theorem and related results as special cases, and provide a characterization in terms of empirically-estimable elasticities much like those in the nonlinear income tax derivation of Saez (2001). Unlike much of the existing literature on commodity taxation, we provide a characterization of optimal tax systems that implement the optimal mechanism. Additionally, we characterize simpler tax systems, such as those that involve a nonlinear earnings tax and a separable tax on capital income (either linear or nonlinear), or those that involve a nonlinear earnings tax and an earnings-dependent but otherwise linear capital income tax.

Our model generalizes the model of Saez (2002), where consumers with heterogeneous earning abilities and tastes choose labor supply and a consumption bundle that exhausts their after-tax income. For concreteness, we describe the consumption bundle as consisting of consumption and savings. The policymaker chooses a flexible nonlinear tax system that depends on both earnings and savings (non-dependence on consumption is without loss of generality) to maximize a (weighted) utilitarian aggregation of individuals' utilities. As in Mirrlees (1976), the policymaker does not observe earnings abilities or tastes for savings, and thus is restricted to second-best policy tools that must trade off the policymaker's redistributive goals against the distortionary effects on labor supply and savings. In the absence of any restrictions on the policymaker's choice of tax systems, our model matches the mechanism design framework of Golosov et al. (2013).

We establish the following six key results in this framework.

First, we show that under mild regularity assumptions, the optimal mechanism of Golosov et al. (2013) can be implemented by a smooth tax system. This allows us to then characterize the optimal

[1] See also Konishi (1995).

[2] Saez (2002) answered the qualitative question of when a "small" commodity tax can increase welfare in the presence of preference heterogeneity, but left to future work the task of deriving an expression for the optimal tax, writing "It would of course be extremely useful to obtain optimal commodity tax formulas" in such a general framework.

mechanism not in terms of unobserved structural primitives, as in Golosov et al. (2013), but instead in terms of observable tax elasticities, as in Saez (2001).

Our second key result is then a characterization of optimal smooth tax systems. We show that optimal earnings tax rates take a form much like that in Saez (2001). Optimal savings tax rates depend on a key sufficient statistic for preference heterogeneity: the difference between the cross-sectional variation of savings s with earnings z, $s'(z)$, and the causal effect of income windfalls on savings, which we denote $s'_{inc}(z)$. The residual, $s'_{pref}(z) := s'(z) - s'_{inc}(z)$ is the sufficient statistic for preference heterogeneity. We show how this statistic can be estimated from existing empirical data and from behavioral responses to policy reforms, avoiding the need for explicitly modeling the relationship between unobserved preferences and ability. The optimal savings tax rate formula takes a form much like that of the optimal earnings tax rate, except that it is directly propotional to $s'_{pref}(z)$, that earnings z are replaced by savings s , and with the earnings elasticity replaced by the elasticity of savings with respect to the savings tax rate.[3] This simple sufficient statistics formulation provides an immediate generalization of the Atkinson-Stiglitz Theorem, as it implies that the optimal savings tax rate is everywhere zero when $s'_{pref}(z) = 0$ for all earnings levels z.

In the second part of the paper, we explore what we call simple tax systems. We show that across a large number of countries, most taxes on various savings vehicles can be classified as one of three types: (i) a separable linear (SL) savings tax, as in Saez (2002) (ii) a separable nonlinear (SN) savings tax, and (iii) a system with a linear earnings-dependent (LED) savings tax, which allows, e.g., for lower-income people to have their savings taxed at a lower rate, as is the case for Long-Term Capital Gains in the U.S.

Our third contribution is to show that for all three of these simple tax systems, the optimal policy can be expressed using a sufficient statistics formula like that of the optimal smooth tax system. In fact, the formulas for the savings tax rates in the SN and LED systems are identical to the formula derived for the optimal smooth tax system. The formula for the optimal savings tax rate in an SL system is necessarily different, but still retains a similar form. These formulas are written in a form that also allows for interpretation as Pareto efficiency conditions, so that one can test whether an existing tax system is consistent (in the sense of Pareto efficiency) with the prevailing nonlinear income tax.

Our fourth contribution is to show that under a narrower—but still surprisingly general—set of assumptions, the SN and LED systems are capable of implementing the *optimal* mechanism. This suggests that the simple types of tax systems found frequently across the world allow for sufficient policy flexibility to achieve the welfare gains available even under much more complicated systems. And the methods we develop provide a practical and portable methodology for studying the optimal form of these tax systems using realistically available data.

Fifth, we show that our main results about SL, SN, and LED systems generalize tractably to multidimensional heterogeneity and to a potentially suboptimal income tax. Even under these more general assumptions, the causal effect of income on savings, together with the cross-sectional profile

[3]In the special case where $s'_{pref}(z) = s(z)$, our formula coincides with the formula of Saez and Stantcheva (2018).

of savings across the income distribution, are together sufficient statistics for characterizing optimal savings tax rates. We also extend our analysis to incorporate government motives to encourage savings. Prominent examples include misalignment in the weight attached to future generations in the context of bequest taxation (e.g. Farhi and Werning, 2010), or agents' present focus in the context of savings taxation (e.g. Moser and Olea de Souza e Silva, 2019; Lockwood, 2020).

Sixth, we use our sufficient statistics formulas to study the optimal savings tax. To obtain our key sufficient statistics, we use the Distributional National Accounts micro-files of Piketty et al. (2018) to calibrate the distribution of savings across the income distribution; we use recent estimates of the medium-run marginal propensity to consume (MPC) of Fagereng et al. (2019) and others to estimate the causal effect of income on savings; we use recent evidence from Agersnap and Zidar (2020) and Jakobsen et al. (2020) to obtain a range of estimates for the savings elasticity; and we use the meta-analysis of Chetty (2012) for the earnings elasticity. These estimates seem to imply a quantitatively important role for preference heterogeneity, which leads us to compute savings tax rates that become substantial at high incomes, with a highly progressive schedule. We find that the savings tax at the low-end of the income distribution is zero or slightly negative, because savings are approximately stable at zero across low incomes. However, savings rates increase with income, outpacing estimates of the marginal propensity to save out of additional income. This suggests that observed savings heterogeneity cannot be fully explained by causal income effects, implying that $s'_{pref}(z) > 0$, and thus that savings taxes are an efficient means of redistribution at higher incomes.

Our estimates of optimal savings tax rates are notably higher than those currently in place in the U.S., as well as those obtained in prior work. For example, the literature on dynamic taxation, which assumes homogeneous preferences but derives a theoretically robust role for capital taxation via the inverse Euler equation (e.g., Golosov et al., 2003; Golosov and Tsyvinski, 2006), tends to find optimal savings "wedges" of only several percentage points (see, e.g., Golosov and Tsyvinski, 2015; Golosov et al., 2016; Farhi and Werning, 2013). Our (preliminary) findings suggests that preference heterogeneity in savings tax rates may play a quantitatively much larger role in optimal savings tax policy than do the social insurance motives analyzed in the dynamic taxation work. Moreover, we also find higher optimal savings tax rates than previous work studying preference heterogeneity using a structural approach; see Golosov et al. (2013). One possible reason is that Golosov et al. (2013) study preference heterogeneity by regressing a structural estimate of time preferences on a plausibly noisy proxy of earnings ability (performance on the Armed Forces Qualification Test), which may lead attenuation bias due to a noisy right-hand-side variable. Our directly implementable sufficient statistics formulas provide a complementary approach to quantifying preferences heterogeneity, without the need for structural estimates of time preferences, or the need to rely on possibly noisy proxies for earnings ability.

Our paper contributes most directly to the literature studying optimal commodity and savings taxes in the presence of correlated preference heterogeneity. Saez (2002) derives the conditions under which the optimal linear commodity tax is strictly positive or negative (thus departing from the benchmark case of the Atkinson-Stiglitz Theorem). More recently, Allcott et al. (2019) derive a sufficient statistics formula for the optimal linear commodity tax in the presence of nonlinear income taxation and correlated preference heterogeneity. Relative to Allcott et al. (2019), we characterize a much broader class of tax systems with varying degrees of simplicity, including some that can implement the allocation from the optimal mechanism.

This paper also contributes to the literature characterizing optimal taxes on capital, inheritances, and estates in the presence of correlated preference heterogeneity. Golosov et al. (2013) derives conditions characterizing the optimal mechanism in a model like the one we study—these conditions on structural "wedges" are those that would be implemented by the sufficient statistics conditions we present in Section 3.3. In parallel work Gerritsen et al. (2020) study necessary conditions for the optimal capital tax in the presence of a different type of correlated heterogeneity—heterogeneous rates of return—which also generates an expression for optimal marginal capital tax rates akin to the wedges in Golosov et al. (2013) and the marginal tax rates in Section 3.3. Our paper is complementary, providing a strategy for implementing these formula from statistics that can be estimated in empirical data (Section 3.2), and conditions under which these formulas are not only necessary, but also sufficient to implement the optimal mechanism. Moreover, we show that our sufficient statistic results can be adapted to encompass such type of correlated heterogeneity that are not driven by intrinsic preferences, thereby providing a unifying framework.

Finally, our work relates to the larger work on optimal taxation of capital. The dynamic taxation literature establishes a robust role for savings taxation through the inverse Euler equation, but assumes that savings preferences do not vary with earnings ability. Our work is complementary in relaxing the assumption of homogeneous savings preferences, but limiting to a more static framework. We believe that extensions of our variational methods could be fruitfully applied to more dynamic models. Our work also complements the recent work of Saez and Stantcheva (2018), who derive optimal capital tax formulas by "putting wealth in the utility function." Mathematically, their baseline results and model resemble the special case of our framework where all differences in savings rates arise from preference heterogeneity.

The rest of this paper proceeds as follows. Section 2 presents our model and assumptions. Section 3 shows that smooth tax systems can implement the optimal mechanism, and provides sufficient statistics for optimal smooth tax systems. Section 4 turns to simple tax systems, providing conditions under which they implement the optimal mechanism, and providing a sufficient statistics characterization of these systems. Section 5.1 characterizes simple tax systems in the presence of multidimensional preference heterogeneity, while also relaxing the assumption that the income tax is set optimally. Section 6 applies our theoretical results to characterize optimal savings tax rates in the U.S.—this section is still preliminary and in progress. Section 7 concludes. All proofs are gathered in the Appendix.

2 Model and assumptions

Agents We consider a population of heterogeneous agents where θ denotes the vector of agents' characteristics and may encapsulate several characteristics. While Section 5.1 studies the case with multidimensional heterogeneity, the rest of the paper follows Saez (2002) and Golosov et al. (2013) in assuming that all characteristics are perfectly correlated such that heterogeneity is one-dimensional and agents can be indexed along a single dimension. Without loss of generality, we thus assume that $\theta \in \mathbb{R}$, and to simplify some analysis we also make the technical assumption that types are supported on a compact set Θ, with a continuously differentiable distribution $F(\theta)$.

Agents live for two periods. In the first period, agents work and receive earnings z, consume c, and save s for consumption in the second period. In the second period, agents are passive and only consume the savings s they have accumulated.[4] Agents' preferences over allocations (c, s, z) are represented by the general utility function $U(c, s, z; \theta)$ indexed by type θ. We impose the following assumption about the utility function.

Assumption 1. *$U(c, s, z; \theta)$ is twice continuously differentiable, increasing and weakly concave in c and s, and decreasing and strictly concave in z. The first derivatives U'_c and U'_s are bounded.*

For example, a frequently-used functional form (e.g. Saez, 2002; Golosov et al., 2013) involves additively separable utility and heterogeneity in agents' productivity w and discount factor δ:

$$U(c, s, z; \theta) = u(c) + \delta(\theta)u(s) - k(z/w(\theta)), \tag{1}$$

with $u(.)$ the utility from consumption and $k(z/w)$ the disutility to work. In this example, there is preference heterogeneity when the discount factor $\delta(\theta)$ covaries with productivity $w(\theta)$. Our analysis encapsulates this particular case as well as any other type of preference heterogeneity.

Formally, preference heterogeneity relates to individual differences in marginal rates of substitutions between commodities. Let $\mathcal{S}$ be the marginal rate of substitution between consumption c and savings s,

$$\mathcal{S}(c, s, z; \theta) := \frac{U'_s(c, s, z; \theta)}{U'_c(c, s, z; \theta)}, \tag{2}$$

we define preference heterogeneity as follows:

Definition 1. *There exists preference heterogeneity if some agents prefer different consumption-savings bundles conditional on having the same earnings level i.e.*

$$\exists \theta_0, \forall (c, s, z), \ \mathcal{S}'_\theta(c, s, z; \theta_0) \neq 0 \tag{3}$$

Going back to the previous example, if agents with higher earnings have higher savings in part because they have a higher preference for savings, this implies that they have a higher marginal rate of substitution when evaluated at the same allocation.

[4] More generally, these results extend to settings where s is some other commodity, or a bequest to a future generation as in the two-period model studied by **?**.

As a last bit of notation, we similarly let $\mathcal{Z}$ denote the marginal rate of substitution between consumption c and earnings z,

$$\mathcal{Z}(c,s,z;\theta) := \frac{U_z'(c,s,z;\theta)}{U_c'(c,s,z;\theta)}. \tag{4}$$

Government An agent's type θ is private information and cannot be observed by the government, which only observes the distribution of types $F(\theta)$. As a result, the government must design incentive-compatible allocations, or more pragmatically, design a tax system that only depends on the observable variables (c,s,z). Adopting as a normalization that consumption c is untaxed, tax systems can be written as $\mathbb{R}^2 \to \mathbb{R}$ functions of the form $\mathcal{T}(s,z)$.

The government's objective is to maximize a weighted sum of agents' utility,

$$\int_\theta \alpha(\theta) U(c(\theta), s(\theta), z(\theta); \theta) \, dF(\theta), \tag{5}$$

where $\alpha(\theta)$ are type-specific Pareto-weights that capture redistributive motives, and $(c(\theta), s(\theta), z(\theta))$ denotes the allocation assigned to type θ. The government's resource constraint is

$$\int_\theta \mathcal{T}(s(\theta), z(\theta)) \, dF(\theta) \geq E \tag{6}$$

where $E \geq 0$ is an exogenous expenditure requirement.

Without any restrictions on the form of the optimal tax system $\mathcal{T}$, the resulting optimal allocation $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$ must solve the following mechanism design program:

$$\max \int_\theta \alpha(\theta) U(c(\theta), s(\theta), z(\theta); \theta) \, dF(\theta) \tag{7}$$

subject to the resource constraint

$$\int_\theta [z(\theta) - s(\theta) - c(\theta)] \, dF(\theta) \geq E \tag{8}$$

and incentive compatibility constraints

$$\forall \left(\theta, \theta'\right) \in \Theta^2, \; U(c(\theta), s(\theta), z(\theta); \theta) \geq U(c(\theta'), s(\theta'), z(\theta'); \theta) \tag{9}$$

We refer to an allocation $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$ that maximizes (7) subject to (8) and (9) as the *optimal incentive-compatible allocation.*

Discussion While preference heterogeneity is what motivates our analysis, our sufficient statistics characterization of optimal tax systems also apply to sources of correlated heterogeneity that are not driven by intrinsic preferences, such as heterogeneous rates of returns on savings or heterogeneous abilities to relabel labor income as capital gains.[5] Indeed, we show that these sources of correlated heterogeneity can be equivalently embedded in agents' preferences. To see this, note that in our baseline setup with preference heterogeneity, agents' problem is to choose gross earnings z, consumption c, and savings s to maximize $U(c, s, z; \theta)$ subject to the budget constraint $c + s \leq z - \mathcal{T}(s, z)$.

Now, suppose that individuals have heterogeneous rates of returns $r(s, \theta)$ which depend both on their amount of savings as well as on their types. Denoting s the accrued savings including interest and $\mathcal{T}(s, z)$ the tax function, agents' budget constraint is then $c + \frac{s}{1+r(s,\theta)} \leq z - \mathcal{T}(s, z)$. This economy is then equivalent to an economy where individuals do not obtain any interests on savings, i.e. $r(s, \theta) = 0$, and maximize the utility function

$$\tilde{U}(c, s, z; \theta) = U\left(c + \frac{sr(s,\theta)}{1 + r(s,\theta)}, s, z; \theta\right) \tag{10}$$

subject to the budget constraint $c + s \leq z - \mathcal{T}(s, z)$. This is because with a rate of return $r(s, \theta)$, individuals get $\frac{sr(s,\theta)}{1+r(s,\theta)}$ more consumption for the same gross earnings z and the same accrued savings s.

Similarly, suppose that individuals can mask χ units of their labor income as savings, so that the observable pre-tax labor income that they are taxed on is $\tilde{z} = z - \chi$. Individuals then choose gross earnings z, consumption c, final savings s, and a masking level χ to maximize $U(c, s, z, \chi; \theta)$ subject to the budget constraint $c + s \leq z - \mathcal{T}(s, z - \chi)$. Introducing χ directly into the utility function allows us to capture effort, time or psychic costs associated with masking, and a complete inability to mask through sufficiently large costs. This economy is then equivalent to an economy where individuals do not mask their income, i.e. $\chi = 0$, and maximize the utility function

$$\tilde{U}(c, s, \tilde{z}; \theta) = \max_{\chi} U\left(c + \chi, s, \tilde{z} + \chi, \chi; \theta\right) \tag{11}$$

subject to the budget constraint that $c + s \leq \tilde{z} - \mathcal{T}(s, \tilde{z})$. This is because when masking χ units of their labor income as savings, individuals get χ more consumption for the same gross earnings z and the same accrued savings s.[6]

This shows the generality of our sufficient statistics approach which will capture all relevant sources of correlated heterogeneity, including those that are not driven by intrinsic preferences. Interestingly, this also highlights that the source of correlated heterogeneity is not as important as previously thought, and we provide a unifying framework to analyze the impact of correlated

[5] Our mechanism design results no longer apply because these types of correlated heterogeneity are ill-defined in a mechanism design approach.

[6] In the same spirit, agents' rates of return could also be endogenous to the time and effort e put into making good investments, with time and effort costs potentially heterogeneous themselves. The equivalent economy would then be one in which individuals maximize the utility function

$$\tilde{U}(c, s, \tilde{z}; \theta) = \max_{\chi, e} U\left(c + \frac{sr(s,e,\theta)}{1 + r(s,e,\theta)} + \chi, s, \tilde{z}, \chi, e; \theta\right)$$

subject to the budget constraint $c + s \leq \tilde{z} - \mathcal{T}(s, \tilde{z})$.

heterogeneity on optimal tax systems.

3 Optimal smooth tax systems

In this section, we provide two key results about *smooth tax systems*, by which we mean twice continuously differentiable functions $\mathcal{T} : \mathbb{R}^2 \to \mathbb{R}$. First, we show that the optimal incentive-compatible allocation is implementable by a smooth tax system under some basic regularity conditions. Second, we leverage our first result to derive a sufficient statistics characterization of optimal smooth tax systems.

This second result constitutes an important advance relative to previous characterizations of the optimal allocation in terms of structural parameters, such as Golosov et al. (2013), as it allows us to characterize optimal tax policy in terms of empirically-estimable elasticities. In particular, our formulas for optimal tax policy do not require structural assumptions and direct measures of how savings preferences vary with earnings ability. Instead, we show that the relationship between the causal effect of earnings on savings, together with the cross-sectional variation of savings with income, provide a sufficient statistic for the preference heterogeneity relevant for desigining optimal taxation of savings, capital income, or bequests.

We derive these two results under the following assumptions, which we maintain throughout the rest of our analysis.

Assumption 2. *Under the optimal incentive-compatible allocation $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$, c^*, s^*, and z^* are smooth functions of θ; s^* and z^* are strictly increasing in θ; any type θ strictly prefers its allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$ to the allocation $(c^*(\theta'), s^*(\theta'), z^*(\theta'))$ of another type $\theta' \neq \theta$.*

Assumptions of smoothness and monotonicity together with strict dominance enable us to derive implementability results as well as to apply variational methods and obtain optimal tax formulas in terms of empirically estimable sufficient statistics. Moreover, the fact that $s^*(\theta)$ and $z^*(\theta)$ are strictly increasing in θ guarantees the existence of an increasing mapping $s^*(z)$, which denotes the savings level s associated with the earnings level z at the optimal incentive-compatible allocation.

The existence of the increasing mapping between types θ and earnings $z^*(\theta)$ requires preference heterogeneity to be bounded. In standard optimal tax frameworks, the Spence-Mirrlees single crossing condition (i.e. $\mathcal{Z}'_\theta(c, s, z; \theta) \geq 0$) usually ensures that more productive agents choose higher earnings. Yet, with preference heterogeneity, this condition is no longer sufficient: more productive agents may actually end up choosing lower earnings if they happen to have a very large aversion to savings. We thus impose a lower bound on preference heterogeneity for commodity s across types θ at the optimal allocation, meaning that $\mathcal{S}'_\theta$ cannot be too negative.[7]

[7] In the case of savings we assume strictly increasing $s^*(\theta)$ (and thus strictly increasing $s^*(z)$), but our results also extend to other commodities with strictly decreasing $s^*(\theta)$ (and thus strictly decreasing $s^*(z)$). The extended Spence-Mirlees condition would then be an upper bound on preference heterogeneity for commodity s across types θ.

Assumption 3. *Under the optimal incentive-compatible allocation* $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$ *, the following extended Spence-Mirrlees condition holds for any* θ*:*

$$\mathcal{Z}'_\theta(c^*(\theta), s^*(\theta), z^*(\theta); \theta) + s^{*\prime}(z^*(\theta))\, \mathcal{S}'_\theta(c^*(\theta), s^*(\theta), z^*(\theta); \theta) \geq 0. \tag{12}$$

3.1 Implementability with smooth tax systems

Definition 2. *We say that an allocation* $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta$ *is implementable with a tax system* $\mathcal{T}$ *if*

1. $\mathcal{T}$ *satisfies type-specific feasibility:* $c(\theta) + s(\theta) + \mathcal{T}(s(\theta), z(\theta)) = z(\theta)$ *for all* $\theta \in \Theta$*, and*
2. $\mathcal{T}$ *satisfies individual optimization:* $(c(\theta), s(\theta), z(\theta))$ *maximizes* $U(c, s, z; \theta)$ *for all* $\theta \in \Theta$*, subject to the constraint* $c + s + \mathcal{T}(s, z) \leq z$*.*

Our first result shows that the optimal incentive-compatible allocation is implementable by some smooth tax system.

Proposition 1. *Suppose that assumptions 1, 2 and 3 hold. Then the optimal incentive-compatible allocation is implementable by a smooth tax system. In this smooth tax system, agents' choices are interior (first-order conditions hold), and their local optima are strict (strict second-order conditions).*

Although it is clear that the optimal incentive-compatible allocation $\{(c(\theta), s(\theta), z(\theta))\}_\theta$ can always be implemented by *some* two-dimensional tax system—e.g., by defining $\mathcal{T}(s(\theta), z(\theta)) = z(\theta) - c(\theta) - s(\theta)$ for each θ, and $\mathcal{T}(s, z) = \infty$ for all other combinations of s and z—such a tax system is not guaranteed to be smooth. A smooth tax system tightens the incentive compatibility constraints in Equation (9), as it must allow agents to independently adjust s and z locally, to points not chosen by any other type, and it is not obvious that these tightened constraints will continue to respect incentive compatibility.

To see this, note that starting from any given allocation $\mathcal{A} = \{(c(\theta), s(\theta), z(\theta))\}_\theta\}_\theta$, a smooth tax system can implement it only by satisfing the first-order conditions

$$\mathcal{T}'_s(s(z(\theta)), z(\theta)) = \mathcal{S}(c(\theta), s(\theta), z(\theta); \theta) - 1 \tag{13}$$

$$\mathcal{T}'_z(s(z(\theta)), z(\theta)) = \mathcal{Z}(c(\theta), s(\theta), z(\theta); \theta) + 1. \tag{14}$$

In the presence of preference heterogeneity, individuals' incentive to deviate from their assigned allocation $(c(\theta), s(\theta), z(\theta))$ are higher under a smooth tax system than under an optimal incentive-compatible mechanism. For example, suppose that higher types θ have a higher taste for savings. If they deviate downward to some other earnings level $z(\theta') < z(\theta)$, then under the optimal mechanism they will be forced to choose savings level $s(\theta')$. Under a smooth tax system, however, the deviating type θ will choose a higher savings level $s' > s(\theta')$ at earnings level $z(\theta')$, making the appeal of deviation higher.

Despite this, in Appendix B.1 we provide a constructive proof that there exists a smooth tax system that does ensure that it is globally optimal for each agent to stick to their assigned bundle $(c(\theta), s(\theta), z(\theta))$. The idea behind the proof is to construct $\mathcal{T}$ such that it satisfies type-specific feasibility and the first-order conditions above, and such that it is sufficiently convex in the savings choice to ensure that any type who chooses a level of earnings $z = z(\theta)$ must prefer to choose a level of savings that is "sufficiently close" to the level of savings $s(\theta)$. This ensures that the set of potential deviations available to a type θ under the smooth tax system is "sufficiently close" to the set of available deviations in the optimal mechanism.

Moreover, the fact that agent's choices satisfy first-order conditions, and satisfy second-order conditions strictly at the optimal incentive-compatible allocation, implies that we can use variational methods to characterize the optimal tax system. This allows us to derive optimal tax formulas expressed in terms of sufficient statistics that transparently highlight the economic forces at play, and more specifically underline the importance of the new sufficient statistic measuring preference heterogeneity.

3.2 Sufficient statistics for smooth tax systems

3.2.1 Definitions

We adopt the usage from Chetty (2009): sufficient statistics are high-level elasticities (or other "program-evaluation estimates") that can be estimated empirically. In our setting these are population statistics quantifying the relationship between savings and earnings, and measurable behavioral responses to tax reforms. To define these statistics, it is helpful to write agents' optimization problem under a tax system $\mathcal{T}(s, z)$ as

$$\max_{z} \left\{ \max_{c,s} U(c, s, z; \theta) \text{ s.t. } c \leq z - s - \mathcal{T}(s, z) \right\} \tag{15}$$

where the inner problem represents the optimal choices of consumption $c(z; \theta)$ and savings $s(z; \theta)$ for a given earnings level z, and the outer problem represents the optimal choice of earnings $z(\theta)$ taking into account endogenous consumption and savings choices.

Earnings responses to tax reforms are captured through $\zeta_z^c(\theta)$, the compensated elasticity of labor income with respect to the marginal labor income tax rate, and $\eta_z(\theta)$, the income effect parameter. Formally,

$$\zeta_z^c(\theta) := -\frac{1 - \mathcal{T}_z'(\theta)}{z(\theta)} \frac{\partial z(\theta)}{\partial \mathcal{T}_z'(\theta)}$$
$$\eta_z(\theta) := -(1 - \mathcal{T}_z'(\theta)) \frac{\partial z(\theta)}{\partial \mathcal{T}(\theta)}$$

where $\mathcal{T}(\theta) := \mathcal{T}(s(z(\theta); \theta), z(\theta))$ is the tax liability, and $\mathcal{T}_z'(\theta) := \frac{\partial \mathcal{T}(s(z(\theta);\theta), z(\theta))}{\partial z}$ is the marginal labor income tax rate of an agent of type θ. Since the earnings choice that pins down $z(\theta)$ takes into account endogenous consumption and savings choices, these elasticity concepts take into account the full sequence of adjustments due to changes in consumption and savings choices, as well as those due to any nonlinearities in the tax system.[8]

Savings responses to tax reforms are captured through $\zeta^c_{s|z}(\theta)$, the compensated elasticity of savings with respect to the marginal savings tax rate, and $\eta_{s|z}(\theta)$, the income effect parameter:

$$\zeta^c_{s|z}(\theta) := -\frac{1+\mathcal{T}'_s(s(z;\theta),z)}{s(z;\theta)} \frac{\partial s(z;\theta)}{\partial \mathcal{T}'_s(s(z;\theta),z)}\Big|_{z=z(\theta)}$$
$$\eta_{s|z}(\theta) := -\left(1+\mathcal{T}'_s(s(z;\theta),z)\right) \frac{\partial s(z;\theta)}{\partial \mathcal{T}(s(z;\theta),z)}\Big|_{z=z(\theta)}$$

where $\mathcal{T}(s(z;\theta),z)$ is the tax liability, and $\mathcal{T}'_s(s(z;\theta),z)$ is the marginal savings tax rate of an agent of type θ who earns labor income z. These elasticity concepts are conditional on z. They measure responses of consumption and savings to tax reforms and nonlinearities in the tax system, holding labor income z fixed.

Correlated preference heterogeneity in consumption-savings choices is captured through $s'_{pref}(\theta)$, which measures the difference between the cross-sectional variation of savings along the earnings distribution and individuals' savings responses to changes in earnings. Formally,

$$s'_{pref}(\theta) := \Bigg(\underbrace{\frac{ds(z;\theta)}{dz}}_{s'(z)} - \underbrace{\frac{\partial s(z;\theta)}{\partial z}}_{s'_{inc}(z)}\Bigg)\Big|_{z=z(\theta)}$$

where $s'(z)$ measures cross-sectional changes in savings moving across agents' types, while $s'_{inc}(z)$ measures individual changes in savings for a given agent's type.

Finally, to encode the policymaker's redistributive objective, we define $\hat{g}(\theta)$ the social marginal welfare weights augmented with the fiscal impact of income effects. These capture the social value of marginally increasing the disposable income of agents θ. Formally,

$$\hat{g}(\theta) := \frac{\alpha(\theta)}{\lambda} U'_c(c(\theta),s(\theta),z(\theta);\theta) + \mathcal{T}'_z(s(\theta),z(\theta)) \frac{\eta_z(\theta)}{1-\mathcal{T}'_z(s(\theta),z(\theta))}$$
$$+ \mathcal{T}'_s(s(\theta),z(\theta)) \left(\frac{\eta_{s|z}(\theta)}{1+\mathcal{T}'_s(s(\theta),z(\theta))} + s'_{inc}(z(\theta)) \frac{\eta_z(\theta)}{1-\mathcal{T}'_z(s(\theta),z(\theta))}\right),$$

where the last term captures the fact that income effects on earnings proportional to $\eta_z(\theta)$ induce changes in savings proportional to $s'_{inc}(z(\theta))$ affecting savings tax revenues.

Moving on to measurement issues and the characterization of optimal smooth tax systems in terms of empirically estimable sufficient statistics, we leverage the one-to-one mapping between agents types θ and agents' earnings z to index all sufficient statistics by earnings z. Similarly, we denote $c(z)$ and $s(z)$ the consumption and savings levels chosen by an agent of type $\theta(z)$ with earnings z.

[8]This corresponds to the type of circular adjustment process described in e.g. Jacquet and Lehmann (2020).

3.2.2 Measurement

Elasticities capturing earnings and savings responses to tax reforms are commonly estimated in the empirical literature, and can be used to obtain the key preference heterogeneity parameter $s'_{pref}(z)$. Because $s'(z)$ is the cross-sectional variation of savings along the income distribution, it can be directly observed from survey or administrative data featuring both earnings and savings. The result below shows how $s'_{inc}(z)$ can be measured in the data.

Proposition 2. *The sufficient statistic $s'_{inc}(z)$ can be measured as follows.*

- *If agents' preferences are weakly separable and the tax system is separable, $s'_{inc}(z) = \frac{1-T'_z(z)}{1+T'_s(s(z))}\eta_{s|z}(z)$*
- *If agents wage rates w and hours h are observable, $s'_{inc}(z) = s(z)\frac{\xi^s_w(z)}{w(z)+h(z)\xi^h_w(z)}$*
- *If we can measure savings and earnings changes in response to tax reforms, $s'_{inc}(z) = \frac{s(z)}{z}\frac{\chi^c_s(z)}{\zeta^c_z(z)}$*

where $\xi^s_w(z)$ is the elasticity of savings with respect to the wage rate, $\xi^h_w(z)$ is the elasticity of hours with respect to the wage rate, and $\chi^c_s(z)$ is the elasticity of savings with respect the marginal net of tax rate on labor income.

This proposition shows how $s'_{inc}(z)$ can be related to empirically estimable elasticity concepts. If, as in example (1) above, agents' preferences are weakly separable between the utility of consumption and savings on the one hand, and the disutility of labor supply on the other hand, then $s'_{inc}(z)$ is proportional to the causal income effect—it is for this reason that we use the subscript *inc*. Intuitively, this is because savings then only depend on earnings through disposable income, assuming the tax system is separable such that the savings tax rate is independent of earnings.

If agents' preferences are not weakly separable but wage rates w and hours h are observable, we can use the fact that labor income is given by the product of hours and the wage rate $z = h \cdot w$, and leverage wage changes to measure $s'_{inc}(z)$. In that case, $s'_{inc}(z)$ can be related to the elasticity of savings with respect to the wage rate and to the elasticity of hours with respect to the wage rate.

Last, if we can measure both earnings and savings changes in response to tax reforms, and more specifically in response to a change in the marginal tax rate on labor income, $s'_{inc}(z)$ can be recovered from the ratio of savings responses to labor income responses. The intuition is that savings responses to this type of tax reform are induced by changes in labor income.

3.3 Sufficient statistics characterization of optimal smooth tax systems

A key result for the sufficient statistics characterizations we provide is the following equivalence lemma for savings tax reforms. This result is a generalization of Lemma 1 from Saez (2002) to a broader class of tax systems. It quantifies the change in earnings induced by a savings tax reform by characterizing the labor income tax reform inducing the exact same change in earnings.[9]

[9] We use the terms "income tax" and "earnings tax" interchangeably to refer to the tax on *labor* income z, as distinct from a tax on savings s, which in some settings can be interpreted as a tax on capital income—see the discussion of their relation in Section 4.

Lemma 1. *A small increase $d\tau_s$ in the marginal savings tax rate faced by agent θ at earnings z, induces the same earnings change as a small increase $s'_{inc}(z)\, d\tau_s$ in the marginal earnings tax rate.*

Intuitively, an agent who reduces earnings by dz reduces savings by $s'_{inc}dz$. The Lemma shows that a $d\tau_s$ increase in the marginal savings tax generates a similar reduction in earnings as a $s'_{inc}d\tau_s$ increase in the earnings tax rate.

Lemma 1 relates the labor supply distortions induced by savings taxes to the labor supply distortions induced by income taxes. Specifically, it shows that increasing savings tax rates will be more distortionary when $s'_{inc}(z)$ is higher. We use this result to provide the following characterization of the optimal marginal income and savings tax rates:

Proposition 3. *An optimal smooth tax system satisfies, at each bundle $(c(z), s(z), z)$ chosen by a type $\theta(z)$, the following marginal earnings tax rate condition*

$$\frac{\mathcal{T}'_z\,(s(z),z)}{1-\mathcal{T}'_z\,(s(z),z)} = \frac{1}{\zeta^c_z(z)}\frac{1}{z\,h_z(z)}\int_{x\geq z}(1-\hat{g}(x))\,dH_z(x) - s'_{inc}(z)\frac{\mathcal{T}'_s\,(s(z),z)}{1-\mathcal{T}'_z\,(s(z),z)} \tag{16}$$

and the following marginal savings tax rate condition

$$\frac{\mathcal{T}'_s(s(z),z)}{1+\mathcal{T}'_s(s(z),z)} = s'_{pref}(z)\frac{1}{\zeta^c_{s|z}(z)}\frac{1}{s\,(z)\,h_z(z)}\int_{x\geq z}(1-\hat{g}(x))\,dH_z(x) \tag{17}$$

Condition (16) on the optimal earnings tax rates constitutes a familiar condition analogous to Saez (2001), with one modification. Because there is a non-zero savings tax rate, the formula also accounts for the distortionary effects on savings caused by distortions to earnings levels.

Condition (17) presents a transparent generalization of the Atkinson-Stiglitz Theorem. When the sufficient statistic for preference heterogeneity, s'_{pref}, is equal to zero, the condition implies that the optimal savings tax rate must equal zero as well. When the statistic is $s'_{pref} > 0$, implying that higher earners have a higher taste for savings, the condition implies that the optimal savings tax rate must be positive.

Interestingly, the optimal savings tax rates satisfy a formula that is remarkably similar to the standard ABC formula for optimal income tax rates. When $s'_{pref} > 0$, the magnitude of the optimal savings tax rate at point (s^*, z^*) is decreasing in the elasticity of savings with respect to the tax rate, increasing in the strength of redistributive motives, and decreasing in the relative density of individuals at point (s^*, z^*).

We can combine conditions (16) and (17) to derive the following Pareto efficiency condition:[10]

Corollary 1. *Any Pareto efficient smooth tax system satisfies, at each bundle $(c(z), s(z), z)$ chosen*

[10] What we here present as a corollary for jointly optimal earnings and savings taxes can also be obtained as a direct result starting from an arbitrary tax system and building Pareto-improving reforms for this tax system.

by a type $\theta(z)$, the following condition

$$\frac{\mathcal{T}_s'(s(z),z)}{1+\mathcal{T}_s'(s(z),z)} = s_{pref}'(z)\frac{\zeta_z^c(z)\,z}{\zeta_{s|z}^c(z)\,s(z)}\frac{\mathcal{T}_z'(s(z),z)+s_{inc}'(z)\mathcal{T}_s'(s(z),z)}{1-\mathcal{T}_z'(s(z),z)} \tag{18}$$

Because the condition in (18) does not feature social marginal welfare weights, it is an efficiency condition that must hold for any tax system that is not Pareto dominated. Intuitively, it quantifies the efficient balance between savings distortions and labor income distortions, given the extent of correlated preference heterogeneity for savings. The larger preference heterogeneity for savings is, the more efficient it is to tax savings relative to earnings at the margin.

An implication of Corollary 1 is that in the absence of any preference heterogeneity, positive savings tax rates are Pareto dominated, providing an extension of the Atkinson-Stiglitz Theorem for nonlinear tax systems. On the other hand, any Pareto efficient tax system must feature nonzero savings tax rates in the presence of preference heterogeneity.

4 Optimal simple savings tax systems

The class of all two-dimensional tax systems $\mathcal{T}(s,z)$ is very diverse, as it allows for tax liabilities to vary across every possible combination of earnings and savings. In practice, tax systems must be defined by policymakers, and implemented by institutions, who face constraints on the degree of complexity that can be accommodated. As a result, real tax functions impose strong restrictions on the set of functions considered. Although the details of these restrictions vary across institutions, most can be classified into a few common sets of functional forms. While these restrictions still allow for substantial flexibility (such as fully nonlinear income taxes) they are nevertheless far more restrictive than the fully general class of two-dimensional functions, and in this sense we consider them "simple." Two natural questions arise. First, does the importance of preference heterogeneity, as measured by s_{pref}', for the design of efficient taxes extends to simple tax systems? Second, how restrictive are simple tax systems i.e. are there circumstances under which the outcomes feasible under the much larger unrestricted class of two-dimensional tax functions can be attained under simple tax systems?

This section answers these questions. We begin by presenting a survey of savings tax policies across a large number of countries, and we identify three common categories of simple tax systems to which most belong: separable linear (SL) savings taxes, separable nonlinear (SN) savings taxes, and linear earnings-dependent (LED) savings taxes. We then derive sufficient statistics formulas characterizing efficient SL, SN, and LED tax systems using the key statistic s_{pref}' to quantify preference heterogeneity. Last, although it is clear that SL tax systems are generally unable to implement the optimal incentive-compatible allocation, due to their inability to satisfy the local savings first-order condition in Equation (13), whether SN and LED systems allow to implement the optimal allocation is not obvious. We conclude this section with a strong result: under weak conditions on the utility function, a SN savings tax *can* implement the optimum (if any smooth tax system can at all). Under somewhat stronger conditions—which we make explicit—a LED savings tax can do the same.

4.1 A taxonomy of common simple savings tax systems

Many governments tax both labor income (earnings) and capital (savings) in some manner. These tax systems take a variety of forms, the details of which depend on the specifics of timing, the nature of the savings vehicle, and many other details. Nevertheless, with some simplifications, many of these tax policies can be interpreted as a $\mathbb{R}^2 \to \mathbb{R}$ function of earnings and savings, analogous to our generalized function $\mathcal{T}(s,z)$. Upon doing so, we observe that nearly all savings tax policies can be categorized as one of three simple types—separable linear (SL), separable nonlinear (SN), and linear income-dependent (LED)—characterized in Table 1.

Type of separable tax system	$\mathcal{T}(s,z)$	$\mathcal{T}'_s(s,z)$	$\mathcal{T}'_z(s,z)$
SL: separable linear	$\tau_s\, s + T_z(z)$	τ_s	$T'_z(z)$
SN: separable nonlinear	$T_s(s) + T_z(z)$	$T'_s(s)$	$T'_z(z)$
LED: linear earnings-dependent	$\tau_s(z)\, s + T_z(z)$	$\tau_s(z)$	$T'_z(z) + \tau'_s(z)\, s$

Table 1: Types of separable tax systems

To translate the highly detail-dependent nature of actual tax codes into something that can be interpreted as $\mathcal{T}(s,z)$, we impose a few important simplifications. First, we treat ordinary income as consisting primarily of labor income (earnings), written as z in our notation. Second, we separately consider taxes on five broad categories of savings vehicles: wealth, capital gains, real property, private pensions, and inheritances.[11] Third, in our model s represents the full amount of resources reserved for future consumption, and so we reinterpret these tax functions (if necessary) as taxes on that basis. Specifically, although taxes on wealth, inheritances, and property are generally written as a tax on the total asset value, taxes on capital gains and pensions are generally written as a tax on the flow of income generated by the underlying asset. As is well appreciated, if one abstracts from heterogeneous or uncertain rates of return (as we do) then one can translate between a tax on wealth and a tax on capital income, with linear taxes remaining linear under this translation.[12]

It turns out that majority of savings tax policies (across countries and across savings vehicles) can be classified as having one of these simple structures. Indeed, each of the three tax systems in Table 1 can for instance be found in the U.S. Most property taxes (levied at the state and local level) take the form of separable linear taxes, with a flat tax rate (independent of one's labor earnings) applied to the assessed value of the total property. The estate tax takes the form of a separable nonlinear tax: the tax rate rises progressively with the value of the estate, but it does not vary

[11] These categories may overlap—real property is a component of wealth, for example—but we use these groups to reflect the tax instruments that many governments use in practice.

[12] For example, if the baseline amount of savings is s and the homogenous, deterministic gross rate of return is $1+r$, then a tax rate τ applied to total savings $(1+r)s$ is equivalent to a tax rate of $\tau(1+1/r)$ applied to capital income rs.

with labor income of the donor or the recipient. Finally, taxes on long-term capital gains and on qualified dividends both take the form of linear earnings-dependent taxes: in 2020, for example, an individual with \$50,000 in labor earnings faced a long-term capital gains tax rate of 15%, whereas an individual earning \$500,000 faced a rate of 20%.

In Table 2, we categorize the tax policies on each class of savings vehicle for 21 countries, most of which fit one of the three simple tax system types from Table 1. In cases where there is some ambiguity (such as the distinction between short-term and long-term capital gains in the United States) we provide supplementary information in Appendix D. Given the ubiquity of these simple tax systems, we next turn to a sufficient statistics characterization of efficient simple tax systems.

4.2 When are simple savings tax systems efficient?

We now derive Pareto-efficiency conditions for SL, SN and LED tax systems which can be directly taken to the data since they only involve observable and measurable statistics. They overall highlight that s'_{pref} remains the key sufficient statistic to determine whether savings should be taxed.

Proposition 4. *Any Pareto efficient simple tax systems which implements a smooth allocation, wherein agents' optima are unique and agents' first-order and second-order conditions strictly hold, must satisfy the following conditions.*

For a SL tax system written as $\mathcal{T}(s,z) = \tau_s s + T_z(z)$*, it must be that,*

$$\frac{\tau_s}{1+\tau_s} = \frac{1}{\int_z \zeta^c_{s|z}(z) s(z)\, dH_z(z)} \int_z s'_{pref}(z)\, \zeta^c_z(z)\, z\, \frac{T'_z(z) + s'_{inc}(z)\tau_s}{1 - T'_z(z)}\, dH_z(z)\,. \tag{19}$$

For a SN tax system written as $\mathcal{T}(s,z) = T_s(s) + T_z(z)$*, it must be that at each earnings* z*,*

$$\frac{T'_s(s(z))}{1 + T'_s(s(z))} = s'_{pref}(z)\, \frac{\zeta^c_z(z) z}{\zeta^c_{s|z}(z) s(z)}\, \frac{T'_z(z) + s'_{inc}(z) T'_s(s(z))}{1 - T'_z(z)}. \tag{20}$$

For a LED tax system written as $\mathcal{T}(s,z) = \tau_s(z)\, s + T_z(z)$*, it must be that at each earnings* z*,*

$$\frac{\tau_s(z)}{1+\tau_s(z)} = s'_{pref}(z)\, \frac{\zeta^c_z(z) z}{\zeta^c_{s|z}(z) s(z)}\, \frac{T'_z(z) + \tau'_s(z)\, s(z) + s'_{inc}(z)\tau_s(z)}{1 - T'_z(z) - \tau'_s(z)\, s(z)}. \tag{21}$$

Pareto-efficiency conditions (20) and (21) for SN and LED tax systems are directly implied by the general Pareto-efficiency condition (18) presented in Corollary 1. The only difference is that these formulas are expressed in terms of the simple savings and earnings tax instruments, lending themselves to empirical applications.

In contrast, the Pareto-efficiency condition (19) for SN tax systems is an independent and novel result. It follows from the same logic but accounts for the fact that the linear savings tax is the same for every agent in the economy. This generates an integral condition, factoring in agents' responses to tax reforms at all earnings levels. As shown in the Appendix, this Pareto-efficiency condition is obtained by combining a reform of the linear savings tax rate with offsetting reforms to the nonlinear earnings tax rate, in the spirit of Konishi (1995), Laroque (2005) and Kaplow (2006).

4.3 When can simple savings tax systems implement the optimum?

In Appendix A.3, we derive sufficient conditions under which the optimal allocation can be implemented by a separable nonlinear (SN) tax system, and by a linear earnings-dependent (LED) tax system. We assume that the optimal allocation satisfies the conditions in Proposition 1, implying that it can be implemented by *some* smooth tax system, so that the question of interest is whether SN or LED systems are among those that implement the optimum. We here provide intuitive descriptions and interpretations of our results, relegating the formal results to the Appendix.

We proceed in three steps. First, we define candidate SN and LED tax systems constructed such as to satisfy type-specific feasibility and individuals' first-order conditions at the optimal allocation, as in Section 3.1. This completely defines candidate tax systems in terms of agents' marginal rates of substitutions $\mathcal{S}$ and $\mathcal{Z}$, and the optimal incentive-compatible allocation $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$. The ability of such simple tax systems to implement the optimum then intuitively depends on the properties of agents' preferences.

Second, we present sufficient conditions under which these candidate SN and LED tax systems also satisfy the local second-order conditions, implying that each type's assigned allocation is a local optimum (Proposition 9). The sufficient conditions for the SN tax system are quite weak ; they are satisfied under many common utility functions used in calibrations of savings and income taxation models, including the simple example function in Equation (1). Beyond conditions related to the concavity of agents' preferences, they involve Spence-Mirrlees conditions for both earnings ($\mathcal{Z}'_\theta \geq 0$) and savings ($\mathcal{S}'_\theta \geq 0$).

The sufficient conditions for the LED tax system are somewhat more restrictive as they place a constraint on the extent of local preference heterogeneity for savings, as compared with preference heterogeneity in earnings. This follows intuitively from the earnings-dependent nature of the linear savings tax. For example, a large positive local preference heterogeneity implies a large positive savings tax rate and thus a large savings tax burden, since the linear tax rate applies to all (including inframarginal) savings. Yet, this large increase in the savings tax liability must then be compensated by a sharp decrease in the earnings tax liability, which may create inconsistencies with agents' earnings choices.

Third, we present sufficient conditions under which local optima are ensured to be global optima for SN and LED tax systems (Proposition 10). These conditions relate to the separability of agents' preferences and further concavity requirements on preferences for consumption and savings. When they are satisfied, the candidate SN and LED systems implement the optimal allocation.

An implication of our results is that under these sufficient conditions, there is an equivalence between SN and LED tax systems, in that they can both implement the optimum. Hence, when we provide sufficient statistics characterization of these tax systems, we are in fact providing a characterization of the same allocation. However, this equivalence relies on the use of different nonlinear earnings tax schedules $T_z(z)$ in each case. If there is an existing nonlinear income tax that cannot be jointly reformed with the savings tax, then the optimal SN and LED savings taxes (conditional on the suboptimal income tax) are generally not equivalent, an issue we investigate in the next section.

5 Extensions

5.1 Multidiminensional heterogeneity and suboptimal income tax

For smooth tax systems, we have shown that the efficiency condition involving our key sufficient statistic for preference heterogeneity arises as a characterizaion of an optimal tax system. Similarly, the efficiency conditions for simple tax systems can be interpreted as pinning down the optimal savings tax schedules, only under the assumption that the nonlinear earnings tax schedule is optimal. In this section, we characterize optimal savings tax schedules in simple tax systems given an existing and potentially suboptimal income tax. Moreover, we extend our analysis to settings with multidimensional heterogeneity.

Formally, we consider settings in which agents are distributed all over the space of savings and earnings (s, z), and we denote $h(s, z)$ the associated probability distribution function. We only assume that this distribution is smooth, but do not otherwise put any restriction on the relationship between earnings and savings. The characterization of optimal savings tax schedules follows as:

Proposition 5. *Consider a given (and potentially suboptimal) earnings tax schedule $T_z(z)$, and assume that under each simple tax system agents first-order and second-order conditions strictly hold.*

In a SL tax system, the optimal linear savings tax rate τ_s satisfies

$$\begin{aligned}&\frac{\tau_s}{1+\tau_s}\int_z\left\{\mathbb{E}\left[s\zeta^c_{s|z}(s,z)\Big|z\right]\right\}dH_z(z)\\&=\int_z\left\{\mathbb{E}\left[(1-\hat{g}(s,z))\,s\Big|z\right]-\mathbb{E}\left[\frac{T_z'(z)+s'_{inc}(s,z)\,\tau_s}{1-T_z'(z)}z\zeta^c_z(s,z)\,s'_{inc}(s,z)\Big|z\right]\right\}dH_z(z)\end{aligned}\tag{22}$$

In a SN system, the optimal nonlinear savings tax $T_s(s)$ satisfies at each level of savings s^*,

$$\begin{aligned}&\frac{T_s'(s^*)}{1+T_s'(s^*)}\int_z\left\{s^*\zeta^c_{s|z}(s^*,z)\right\}h(s^*,z)\,dz\\&=\int_z\left\{\mathbb{E}\Big[1-\hat{g}(s,z)\Big|z,s\geq s^*\Big]\right\}h_z(z)\,dz-\int_z\left\{\frac{T_z'(z)+s'_{inc}(s^*,z)\,T_s'(s^*)}{1-T_z'(z)}z\zeta^c_z(s^*,z)s'_{inc}(s^*,z)\right\}h(s^*,z)\,dz\end{aligned}\tag{23}$$

In a LED system, the optimal linear earnings-dependent savings tax $\tau_s(z)$ satisfies at each level of *earnings* z^*,

$$\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s\Big|z^*\right]h_z(z^*)+\int_{z\geq z^*}\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c(s,z)\Big|z\right]dH_z(z)$$
$$=\int_{z\geq z^*}\left\{\mathbb{E}\Big[(1-\hat{g}(s,z))\,s\Big|z\Big]-\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s_{inc}'(s,z)\Big)\Big|z\right]\right\}dH_z(z) \tag{24}$$

For a SL tax system, Equation (22) characterizes the optimal linear savings tax rate τ_s by considering the impact of a change $d\tau_s$ in the tax rate. This triggers fiscal effects through changes in savings proportional to $\zeta_{s|z}^c(s,z)\,d\tau_s$ and changes in earnings proportional to $\zeta_z^c(s,z)s_{inc}'(s,z)\,d\tau_s$, as well as mechanical effects induced by changes in tax liability proportional to $sd\tau_s$. Aggregating these effects across all savings (through the expectation) and earnings (through the integral) yields the result.

For a SN tax system, Equation (23) pins down the optimal marginal savings tax rate $T_s'(s^*)$. The same economic forces are at play, except that we consider this time a local change $d\tau_s$ of the marginal savings tax rate in a small bandwidth ds of savings around s^*. Fiscal effects are thus local and proportional to the density of agents $h(s^*,z)$, while mechanical effects affect all agents with higher savings levels through changes in tax liability proportional to $d\tau_s ds$. Hence, while this formula involves agents at all earnings levels, it only involves agents with savings higher or equal than s^*.

For a LED tax system, Equation (31) characterizes the optimal linear earnings-dependent savings tax rate $\tau_s(z)$ by considering a change $\delta\tau_s$ in the tax rate phased-in over a bandwidth dz of earnings around z^*. As a result, at earnings z^* the marginal savings tax rate is unchanged, while the marginal earnings tax rate increases by $s(z^*)\,d\tau_s$. Local fiscal effects thus primarily operate through changes in earnings as reflected in the first term. Other terms are extremely similar to the expression characterizing the optimal SL tax system because they relate to the change in the linear savings tax rate above z^*. Hence, while this formula involves agents at all savings levels, it only involves agents with earnings higher or equal than z^*.

The difference between these different types of tax systems is most obvious in the multidimensional case because they impact very different individuals. Yet, these differences also arise in the unidimensional case as can be seen from the unidimensional formulas provided in the Appendix.[13] To see this, consider a fixed (and potentially suboptimal) earnings tax schedule $T_z(z)$. While a SL tax system is obviously a particular case of SN and LED tax systems, the relationship between SN and LED tax systems is less clear. For a given marginal savings tax rate $T_s'(s(z^*))=\tau_s(z^*)$, a LED tax system induces more distortions than a SN tax system at labor income z^* if and only if $\tau_s'(z)\geq 0$. Yet, a LED tax system allows to tax all infra-marginal units of savings at rate $\tau_s(z^*)$, while in a SN tax system the tax on infra-marginal units of savings depends on distortions imposed

[13]We also provide in the Appendix formulas for the optimal earnings tax schedules in each type of simple tax system.

at lower earnings levels.

Last, it is interesting to note that when departing from optimal tax systems by considering potentially suboptimal earnings tax schedules, our key sufficient statistic for preference heterogeneity does not enter optimal savings tax formulas. The intuition is that with suboptimal earnings tax schedules, optimality may prescribe deviating from Pareto-efficiency on equity grounds. For instance, if the earnings tax is much lower than what would be optimal, an inefficiently high savings tax may become optimal on equity grounds as a way to induce higher redistribution: a constrained optimum may not always lie on the Pareto-frontier. In what follows we abstract from these third-best world considerations and rely on Pareto-efficiency conditions to analyze the quantitative implications of preference heterogeneity for the taxation of savings and capital.

5.2 Bequest taxation and behavioral agents

As previously mentioned, our framework can also be interpreted as a bequest model in which parents work and consume in the first period, and leave a bequest to their heirs in the second period. because they are altruistic and value the well-being of future generations. Yet, as pointed out by Farhi and Werning (2010), the weight that parents attach to the well-being of future generations may be too low from a normative perspective. This misalignment in the valuation of future generations introduces a corrective motive for taxation that we now analyze.

We here follow Farhi and Werning (2010) and consider that agents' preferences are additively separable and given by

$$U\big(c, s, z; \theta\big) = u\left(c; \theta\right) - k\left(z; \theta\right) + \beta v\left(s; \theta\right), \tag{25}$$

where $u\left(c;\theta\right)$ is the utility parents derive from consumption c, $k\left(z;\theta\right)$ is the disutility parents incur to obtain earnings z, $v\left(s;\theta\right)$ is the utility heirs derive from a bequest s, and β is the weight parents attach to the well-being of their heirs. Building on Farhi and Werning (2010), we also consider a utilitarian government that maximizes

$$\int_{\theta} \big[U\left(c\left(\theta\right), s\left(\theta\right), z\left(\theta\right); \theta\right) + \nu v\left(s\left(\theta\right); \theta\right)\big] dF\left(\theta\right), \tag{26}$$

where ν parametrizes the degree of misalignment and corresponds to the Lagrange multiplier associated with a constraint that the future generation attains a required level of well-being $\int_{\theta} \nu v\left(s\left(\theta\right)\right) dF\left(\theta\right) \geq \underline{V}$.

In a smooth tax system, optimal bequests taxes are then characterized by the following proposition.

Proposition 6. *In the presence of misalignment on the weight attached to future generations, an optimal smooth tax system satisfies at each earnings z, the following marginal bequest tax rate condition*

$$\frac{\mathcal{T}_s'\left(s(z), z\right)}{1 + \mathcal{T}_s'\left(s\left(z\right), z\right)} = s_{pref}'\left(z\right) \frac{1}{\zeta_{s|z}^{c}(z)} \frac{1}{s(z) h_z(z)} \int_{x \geq z} \left(1 - \hat{g}(x)\right) dH_z(x) - \frac{\nu}{\beta} \hat{g}(z). \tag{27}$$

Moreover, any Pareto-efficient smooth tax system satisfies at each earnings z,

$$\frac{\mathcal{T}_s'\left(s(z),z\right)}{1+\mathcal{T}_s'\left(s\left(z\right),z\right)}+\frac{\nu}{\beta}\hat{g}(z)=s_{pref}'\left(z\right)\frac{\zeta_z^c(z)z}{\zeta_{s|z}^c(z)s(z)}\left[\frac{\mathcal{T}_z'\left(s\left(z\right),z\right)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}+s_{inc}'(z)\left(\frac{\mathcal{T}_s'\left(s(z),z\right)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}+\frac{\nu}{\beta}\hat{g}(z)\right)\right]. \tag{28}$$

Condition (27) pins down the optimal schedule of marginal bequest tax rates. It shows that when the government values more future generations ($\nu > 0$), the presence of misalignment reduces optimal marginal tax rates on bequests. In the absence of preference heterogeneity ($s_{pref}'\left(z\right)=0$), misalignment even calls for marginal bequests subsidies as highlighted by Farhi and Werning (2010). Indeed, misalignment pushes the government to subsidize bequests as a way to incentivize work effort and increase bequests to future generations.

Condition (28) is a Pareto-efficiency condition that pins down the optimal balance between earnings distortions and bequests distortions. As before, the larger preference heterogeneity is, the more bequests should be taxed. Yet, misalignment act as a counteracting force to lower bequests taxes which may call for no or even negative bequests taxes. Since the corrective motive induced by misalignment is proportional to social marginal welfare weights $\hat{g}(z)$, it is larger at low income levels than at high income levels and calls for progressive bequest taxes (Farhi and Werning, 2010).

Interstingly, this model can be equivalently interpreted as our benchmark savings model with present focused individuals. In this interpretation, β is the degree of present focus which could depend on types and thus vary across earnings levels, i.e. $\beta\left(z\right)$. The degree of misalignment would then also be earnings-specific and equal to $\nu\left(z\right)=1-\beta\left(z\right)$, if we were to use as a normative criterion that agents should not discount the future. Since low income individuals tend to exhibit a higher degree of present focus than high income individuals (see e.g. Lockwood, 2020), this would even reinforce the progressivity of the corrective motive and call for even more progressive savings tax rates.

6 Empirical application

To illustrate the quantitative implications of preference heterogeneity for the taxation of savings and capital, we first calibrate the different sufficient statistics from micro data and empirical studies. We then make use of the Pareto-efficiency conditions presented in Proposition 4 to compute the efficient SL, SN and LED savings tax rates, taking the earnings tax schedule as given. Our (preliminary) results show that preference heterogeneity calls for positive and progressive savings tax rates, which range from 0% at the bottom of the income distribution up to 20% at the top in our baseline calibration.

Calibration The earnings and savings distribution in the U.S. is calibrated based on the Distributional National Accounts micro-files of Piketty et al. (2018). We use 2019 measures of pretax income (*ptinc*) and net personal wealth (*hweal*) at the individual level as well as the age category (20 to 44 years old, 45 to 64, and above 65). Discretizing the income distribution into percentiles by age group, our measure of annualized earnings during the working life z at the n-th percentile is constructed by averaging earnings at the n-th percentile across those aged 20 to 44 and those aged 45 to 64. To measure accumulated savings over the working life, our measure of annualized savings s at the nth percentile of income is constructed as the difference between the median wealth within the nth percentile of those aged 45 to 64 and those aged 20 to 44, scaled by the average length of the accumulation period ($54.5 - 32 = 22.5$). This yields a distribution of earnings z and savings $s(z)$, and pins down the cross-sectional variation in savings $s'(z)$.[14]

Figure 3 plots our estimate of savings across the income distribution in the U.S. The figure shows that savings rates are approximately zero at the low end, but increase substantially with income.

To calibrate the causal income effect on savings $s'_{inc}(z)$, we rely on the work of Fagereng et al. (2019) who estimate marginal propensities to consume (MPC) across the earnings distribution using information on lottery prizes linked with administrative data in Norway. Agents' consumption peaks during the winning year and gradually reverts to their previous value afterwards. Over a 5-year horizon, they estimate agents consume close to 90% of the prize, which translates into MPCs of 0.9. They find little evidence of variation across earnings levels. Under the assumption that preferences are weakly separable with respect to the disutility of labor supply, these MPCs estimates map to a constant s'_{inc} parameter of 0.1, which suggests that a large fraction of the observed cross-sectional variation in savings is related to preference heterogeneity.

Because savings rates increase substantially with income, but s'_{inc} does not, we conclude that $s'_{pref}(z)$ increases with income. This will lead our estimates of savings tax rates to be progressive.

For the earnings elasticity we take the standard value $\zeta^c_z = 0.33$ from the meta-analysis of Chetty (2012). The value of the savings elasticity $\zeta^c_{s|z}$ is harder to obtain, in part because different savings vehicles may trigger different responses to taxes and because different time horizons can lead to different results. Recent work on the very wealthy by Agersnap and Zidar (2020) uses state-variation in capital gains tax rates in the U.S. to estimate a policy-relevant elasticity of capital gains realizations with respect to the tax rate over a ten-year period. They report an elasticity estimate of -0.53, which reduces to -0.29 after accounting for the fact that taxpayers' responses are often shifts to other types of assets that inflate other tax bases and lead to additional tax revenues. We show in Appendix C.1 how an elasticity of capital gains realizations of around -0.4 translates into a savings elasticity $\zeta^c_{s|z}$ around 2, which we consider to be a likely loose upper bound on the actual savings tax elasticity. Another recent study by Jakobsen et al. (2020) provides unique evidence on behavioral responses to wealth taxation in Denmark. They estimate a long-run elasticity of taxable wealth with respect to the after-tax rate of return (which can be interpreted as $\zeta^c_{s|z}$) equal to 0.77 for the moderately wealthy and equal to 1.15 for the very wealthy. We assume the savings elasticity $\zeta^c_{s|z}$ is constant across income and report results for elasticity parameters of 0.7, 1 (baseline), and 2.

[14] To calibrate consumption levels, we use post-tax disposable income (*diinc*) net of our measure for annualized savings. See Appendix C.1 for additional information on our calibration procedure.

It is interesting to contrast our sufficient statistics calibration with the structural calibration of Golosov et al. (2013), who measure preference heterogeneity through differences in discount factors across *ability* levels. They infer discount factors from a simple optimization model applied to survey data on individuals' household income paths and net worth, while they use survey respondents' results to the Armed Forces Qualification Test (AFQT) to build a proxy for ability. Because they do not find significant variations in discount factors across their proxy for ability, their calibration implies very little preference heterogeneity. We show in Appendix C.2 that their calibrated structural model yields an earnings elasticity $\zeta_z^c = 0.33$, a savings elasticity $\zeta_{s|z}^c = 0.52$, and a causal income effect parameter $s'_{inc} = 0.34$.

Results Figure 2 reports estimates of SL, SN and LED savings tax rates that satisfy the Pareto efficiency formulas in Proposition 4, taking the existing U.S. income tax schedule and income distribution as given. Each panel reports results for a different value of the savings elasticity. For SL tax systems, the linear savings tax rate τ_s is by definition constant across earnings levels. For LED tax systems, the linear savings tax rate $\tau_s(z)$ is earnings dependent and we thus report the linear savings tax rate at each earnings level. For SN tax systems, the nonlinear savings tax schedule $T_s(s)$ depends on the value of savings s, and not on earnings z. To facilitate comparisons between the systems, we plot the value of the marginal savings tax rate at the corresponding earnings level, employing our calibrated distribution of savings across earnings $s(z)$.

Because we find that $s'_{pref}(z)$ is positive and increasing in z, we find that savings tax rates are positive and increasing in earnings in the SN and LED tax systems. The magnitudes of course depend on the value of the savings elasticity parameter. In the baseline ($\zeta_{s|z}^c = 1$), savings tax rates in SN and LED tax systems range from 0% for annual incomes below \$50,000, then steadily increase up to 20% for annual incomes around \$200,000, and remain somewhat constant above. The efficient linear savings tax rate in a SL tax system is around 8%. Changing the savings elasticity parameter only shifts the efficient savings tax rates without affecting the overall pattern: preference heterogeneity calls for positive and progressive savings tax rates.

These estimates are substantially higher than the prevailing savings tax rates in the U.S., which are shown in Figure 1, even under our upper bound savings elasticity estimate of 2. As detailed in Appendix D, we base these estimates on the U.S. income tax code, together with a decomposition of savings sources by income provided by Bricker et al. (2019).

7 Conclusion

This paper presents sufficient statistics formulas characterizing the optimal smooth tax system on earnings and savings (or other dimensions of consumption) in the presence of preference heterogeneity that is correlated with earnings ability. The formulas depend on a key sufficient statistic for preference heterogeneity, $s'_{pref}(z)$, which can be estimated from empirical data on cross sectional earnings and savings, and on observed behavioral responses to tax reforms. We show that such a smooth system can implement the optimal mechanism under weak conditions. Under only slightly stronger conditions, the optimal mechanism can also be implemented by two familiar types of "simple" separable tax systems which combine a nonlinear tax on earnings with a nonlinear tax on savings, or with a linear earnings-dependent savings tax. We derive intuitive sufficient statistics formulas for these separable tax systems, as well as for linear separable savings taxes, and we present sufficient conditions under which these necessary conditions are also sufficient to characterize the optimal simple tax systems. Together, these results provide a practical and general method for quantifying optimal tax systems for saving, inheritances, and other commodities in the presence of correlated preference heterogeneity.

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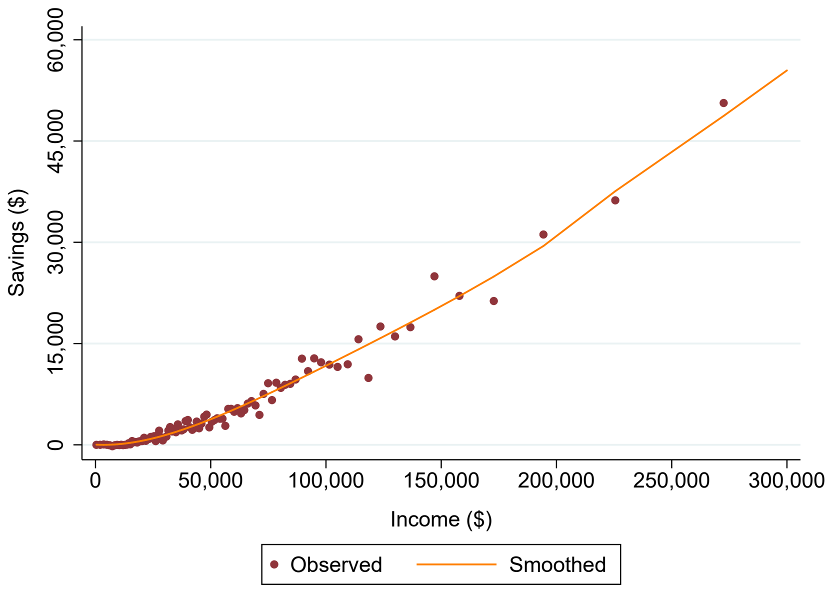
Tables & Figures

Table 2: Tax systems applied to different savings vehicles, by country.

Country	Wealth	Capital Gains	Property	Pensions	Inheritance
Australia	–	Other	SL, SN	SL	–
Austria	–	Other	SL, SN	SN	–
Canada	–	Other	SL	SN	–
Denmark	–	SN	SL, SN	SL, SN	SN
France	–	Other	Other	SL, SN	SN
Germany	–	Other	SL	SN	SN
Ireland	–	SN	SL, SN	SN	SN
Israel	–	Other	Other	SN	–
Italy	SL, SN	SL	SL	SL	SL, SN
Japan	–	SL, SN	SN	SN	SN
Netherlands	SN	SL	SL, SN	SN	SN
New Zealand	–	Other	SN	SL, LED	–
Norway	SN	SL	SL	SN	–
Portugal	–	SL	Other	SN	SL
Singapore	–	Other	SN	SN	–
South Korea	–	SN	SN	SN	SN
Spain	SN	SN	SL, SN	SN	SN
Switzerland	SN	SN	SL, SN	SN	SN
Taiwan	–	SL, SN	SL, SN	SN	SN
United Kingdom	–	Other	SN	SN	SN
United States	–	LED	SL	SN	SN

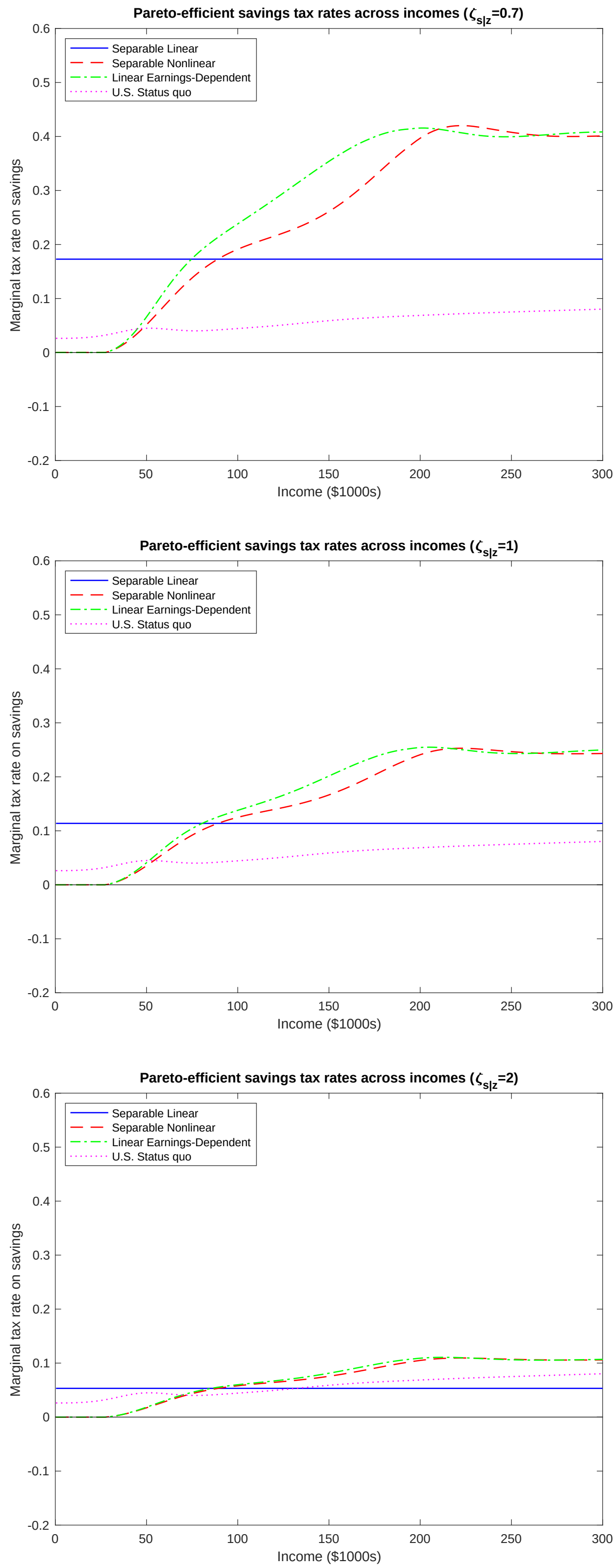
Notes: This table classifies tax systems applied to different savings vehicles across countries in 2020 according to the types in Table 1.

Figure 1: Savings Rates by Income in the United States



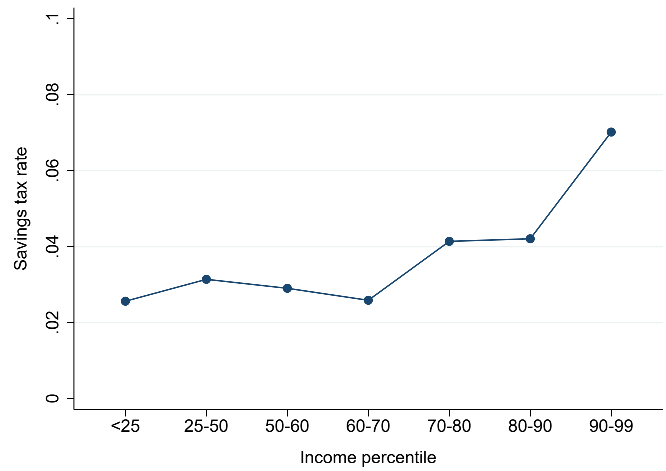
Notes: The earnings and savings distribution in the U.S. is calibrated based on the Distributional National Accounts micro-files of Piketty et al. (2018). We use 2019 measures of pretax income (ptinc) and net personal wealth (hweal) at the individual level as well as the age category (20 to 44 years old, 45 to 64, and above 65). Discretizing the income distribution into percentiles by age group, our measure of annualized earnings during the working life z at the n-th percentile is constructed by averaging earnings at the n-th percentile across those aged 20 to 44 and those aged 45 to 64. To measure accumulated savings over the working life, our measure of annualized savings s at the n-th percentile of income is constructed as the difference between the median wealth of those aged 45 to 64 and those aged 20 to 44, scaled by the average length of the accumulation period $(54.5 - 32 = 22.5)$. This yields a distribution of earnings z and savings $s(z)$, and pins down the cross-sectional variation in savings $s'(z)$. To calibrate consumption levels, we use post-tax disposable income (diinc) net of our measure for annualized savings. See Appendix C.1 for additional information on our calibration procedure.

Figure 2: Savings Tax Rates Implied by Pareto Efficiency Formulas



Notes: This figure presents the savings tax values that satisfy the Pareto efficiency formulas in Proposition 4, plotted against the earnings level to which they apply.

Figure 3: Actual Savings Tax Rates in the United States, by Income Percentile



Notes: This figure provides estimates of the actual savings tax rates in the United States. See Appendix C.3 for details.

Online Appendix

Sufficient Statistics for Nonlinear Tax Systems With Preference Heterogeneity

Antoine Ferey, Benjamin B. Lockwood, and Dmitry Taubinsky

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A Supplementary theoretical results

A.1 Optimal savings tax formulas for simple tax systems

We now present optimal savings tax formulas which characterize the optimal savings tax schedule for *any* given earnings tax schedule—including a potentially suboptimal one.

Proposition 7. *Consider a given (and potentially suboptimal) earnings tax schedule $T_z(z)$, and assume that under each simple tax system agents first-order and second-order conditions strictly hold, and that savings strictly increase with earnings in the cross-section.*
In a SL tax system, the optimal linear savings tax rate τ_s satisfies

$$\frac{\tau_s}{1+\tau_s}\int_z s(z)\zeta^c_{s|z}(z)dH_z(z) = \int_z \left\{(1-\hat{g}(z))\ s\left(z\right) - \frac{T_z'(z)+s'_{inc}(z)\,\tau_s}{1-T_z'\left(z\right)} z\zeta^c_z(z)\, s'_{inc}(z)\right\} dH_z(z) \quad (29)$$

In a SN system, the optimal nonlinear savings tax $T_s\left(s\right)$ satisfies at each level of savings $s\left(z^*\right)$,

$$\frac{T_s'(s(z^*))}{1+T_s'(s(z^*))} s(z^*)\zeta^c_{s|z}(z^*)h_z(z^*)$$
$$= s'\left(z^*\right)\int_{z^*}^{\bar{z}} (1-\hat{g}(z))\, dH_z(z) - \frac{T_z'(z^*)+s'_{inc}(z^*)T_s'(s(z^*))}{1-T_z'(z^*)} z^*\zeta^c_z(z^*) s'_{inc}(z^*)h_z(z^*) \quad (30)$$

In a LED system, the optimal linear earnings-dependent savings tax $\tau_s(z)$ satisfies at each level of earnings z^*,

$$\frac{T_z'\left(z^*\right)+\tau_s'\left(z^*\right)s\left(z^*\right)+s'_{inc}(z^*)\tau_s(z^*)}{1-T_z'\left(z^*\right)-\tau_s'\left(z^*\right)s\left(z^*\right)} z^*\zeta^c_z(z^*)\, s(z^*)\, h_z(z^*) + \int_{z^*}^{\bar{z}} \frac{\tau_s(z)}{1+\tau_s(z)} s(z)\zeta^c_{s|z}(z)\, dH_z(z)$$
$$= \int_{z^*}^{\bar{z}} \left\{(1-\hat{g}(z))\, s\left(z\right) - \frac{T_z'\left(z\right)+\tau_s'\left(z\right)s\left(z\right)+s'_{inc}(z)\tau_s(z)}{1-T_z'\left(z\right)-\tau_s'\left(z\right)s\left(z\right)} z\zeta^c_z(z) s'_{inc}(z)\right\} dH_z(z) \quad (31)$$

These optimal savings tax formulas are all different, reflecting differences between the savings tax instruments that we consider. Yet, they share common elements. First, the preference heterogeneity term $s'_{pref}\left(z\right)$ no longer appears in the formulas. The intuition is that outside of the full optimum, it may still be desirable to tax savings in the absence of preference heterogeneity, implying that optimality may clash with Pareto-efficiency when the earnings tax is suboptimal. Second, $s'_{inc}\left(z\right)$ is a key sufficient statistic for optimal savings tax schedules. Indeed, by Lemma 1, a larger $s'_{inc}\left(z\right)$ means that savings tax reforms impose higher distortions on earnings and thus generally calls for lower savings tax rate.

A.2 Optimal earnings tax formulas for simple tax systems

We now present optimal earnings tax formulas for each type of simple savings tax system. For each type of tax system, this characterizes the optimal earnings tax schedule for *any* given savings tax schedule—including potentially suboptimal ones.

Proposition 8. *Assume that under each simple tax system agents first-order and second-order conditions strictly hold, and that savings strictly increase with earnings in the cross-section.*
In a SL tax system, for a given (potentially suboptimal) linear savings tax rate τ_s the optimal marginal earnings tax rate $T_z'(.)$ satisfies at each level of earnings z^*,

$$\frac{T_z'(z^*)}{1-T_z'(z^*)} = \frac{1}{\zeta_z^c(z^*)}\frac{1}{z^* h_z(z^*)}\int_{z^*}^{\bar{z}} (1-\hat{g}(x))\, dH_z(x) - s_{inc}'(z^*)\frac{\tau_s}{1-T_z'(z^*)} \tag{32}$$

In a SN system, for a given (potentially suboptimal) nonlinear savings tax schedule $T_s(.)$ the optimal marginal earnings tax rate $T_z'(.)$ satisfies at each level of earnings z^*,

$$\frac{T_z'(z^*)}{1-T_z'(z^*)} = \frac{1}{\zeta_z^c(z^*)}\frac{1}{z^* h_z(z^*)}\int_{z^*}^{\bar{z}} (1-\hat{g}(x))\, dH_z(x) - s_{inc}'(z^*)\frac{T_s'(s(z^*))}{1-T_z'(z^*)} \tag{33}$$

In a LED system, for a given (potentially suboptimal) linear earnings-dependent savings tax schedule $\tau_s(.)$ the optimal marginal earnings tax rate $T_z'(.)$ satisfies at each level of earnings z^*,

$$\frac{T_z'(z^*)+\tau_s'(z^*)\,s(z^*)}{1-T_z'(z^*)-\tau_s'(z^*)\,s(z^*)} = \frac{1}{\zeta_z^c(z^*)}\frac{1}{z^* h_z(z^*)}\int_{z^*}^{\bar{z}} (1-\hat{g}(x))\, dH_z(x) - s_{inc}'(z^*)\frac{\tau_s(z^*)}{1-T_z'(z^*)-\tau_s'(z^*)\,s(z^*)} \tag{34}$$

These conditions pinning down optimal marginal earnings tax rates are a direct application of Equation (16) presented in Proposition 3 for smooth tax systems. While formulas for SL and SN tax systems look almost identical to the general condition, the formula for LED tax system looks a bit different. This difference only reflects the fact that for a LED tax system the marginal earnings tax rate is given by $\mathcal{T}_z'(s,z) = T_z'(z) + \tau_s'(z)\,s(z)$, accounting for the earnings-dependent nature of the savings tax instrument.

A.3 Implementation results for simple tax systems

We proceed in three steps to provide sufficient conditions under which some SN and LED tax systems decentralize the optimal incentive-compatible allocation.

First, we define candidate SN and LED tax systems that satisfy type-specific feasibility and agents' first-order conditions at the optimal allocation. Second, in Proposition 9 we present sufficient conditions under which these SN and LED tax systems also satisfy agents' second-order conditions at the optimal allocation, implying local optimality. Third, in Proposition 10, we present sufficient conditions under which local optima are ensured to be global optima, implying that the candidate SN and LED systems are indeed implementing the optimal allocation.

Step 1: Defining candidate tax systems. We first define a candidate SN tax system $\mathcal{T}(s,z) = T_s(s)+T_z(z)$, with the nonlinear functions T_s and T_z defined across all savings and earnings bundles of the optimal allocation $\mathcal{A} = (c^*(\theta), s^*(\theta), z^*(\theta))_\theta$ as follows:

$$T_s(s^*(\theta)) := \int_{\vartheta=\underline{\theta}}^{\theta} \left(U'_s(\vartheta)/U'_c(\vartheta) - 1\right) s^{*\prime}(\vartheta) d\vartheta, \tag{35}$$

$$T_z(z^*(\theta)) := z^*(\underline{\theta}) - s^*(\underline{\theta}) - c^*(\underline{\theta}) + \int_{\vartheta=\underline{\theta}}^{\theta} \left(U'_z(\vartheta)/U'_c(\vartheta) + 1\right) s^{*\prime}(\vartheta) d\vartheta \tag{36}$$

where $\underline{\theta}$ denotes the lowest earning type of the compact type space Θ, and the derivatives are evaluated at the bundle assigned in the optimal allocation (e.g., $U'_s(\vartheta) = U'_s(c^*(\vartheta), s^*(\vartheta), z^*(\vartheta); \vartheta)$). Under this tax system, the optimal allocation satisfies by definition each type's first-order conditions for individual optimization given in Equations (13) and (14). By Lemma 2, this tax system thus satisfies type-specific feasibility.

We similarly define a candidate LED tax system $\mathcal{T}(s,z) = \tau_s(z) \cdot s + T_z(z)$ as follows:

$$\tau_s(z^*(\theta)) := U'_s(\theta)/U'_c(\theta) - 1, \tag{37}$$

$$\begin{aligned} T_z(z^*(\theta)) := & z^*(\underline{\theta}) - s^*(\underline{\theta}) - c^*(\underline{\theta}) \\ & + \int_{\vartheta=\underline{\theta}}^{\theta} \left(U'_z(\vartheta)/U'_c(\vartheta) + 1\right) s^{*\prime}(\vartheta) d\vartheta - s^*(z) \cdot \left(\tau_s(z) - \tau_s(z^*(\underline{\theta}))\right). \end{aligned} \tag{38}$$

This tax system also satisfies local first-order conditions for individual optimization and type-specific feasibility.

Step 2: Local maxima. We can now derive sufficient conditions under which the above candidate SN and LED tax systems satisfy the second-order conditions for individual optimization, implying that under these conditions assigned bundles are local optima . These conditions can be simply stated in terms of the marginal rates of substitution between consumption and, respectively, savings $\mathcal{S}(c,s,z;\theta)$ and earnings $\mathcal{Z}(c,s,z;\theta)$. These marginal rates of substitutions are smooth functions of c, s, z, and θ by the smoothness of the allocation and the utility function, and sufficient conditions for local second-order conditions are given by the following proposition.

Proposition 9. *Suppose that an allocation satisfies the conditions in Proposition 1. Under the SN tax system defined by Equations* (35) *and* (36)*, each agent's assigned choice of savings and earnings is a local optimum if the following conditions hold at each point in the allocation:*

$$\mathcal{S}'_c \geq 0,\ \mathcal{S}'_z \geq 0,\ \mathcal{S}'_\theta \geq 0 \tag{39}$$

and

$$\mathcal{Z}'_c \leq 0,\ \mathcal{Z}'_s \geq 0,\ \mathcal{Z}'_\theta \geq 0. \tag{40}$$

Under the LED tax system defined by Equations (37) *and* (38)*, each agent's assigned choice of savings and earnings is a local optimum if the utility function is additively separable in consumption, savings, and earnings ($U''_{cs} = 0$, $U''_{cz} = 0$, and $U''_{sz} = 0$), and additionally the following conditions* *hold at each point in the allocation:*

$$\mathcal{S}'_\theta \geq 0,\ \mathcal{S}'_\theta \leq \frac{z^{*\prime}(\theta)}{s^{*\prime}(\theta)}\mathcal{Z}'_\theta,\ \mathcal{S}'_\theta \leq s^{*\prime}(\theta)\left(\mathcal{S}\cdot\mathcal{S}'_c - \mathcal{S}'_s\right). \tag{41}$$

The sufficiency conditions (39) and (40) are quite weak; they are satisfied under many common utility functions used in calibrations of savings and income taxation models, including the simple example function in Equation (1). Conditions $\mathcal{S}'_\theta \geq 0$ and $\mathcal{Z}'_\theta \geq 0$ are single crossing conditions for savings and earnings, while other conditions intuitively relate to the concavity of preferences.

The sufficiency conditions for LED systems are more restrictive. Beyond the single crossing conditions $\mathcal{S}'_\theta \geq 0$ and $\mathcal{Z}'_\theta \geq 0$, they place a constraint on the extent of local preference heterogeneity for savings, as compared with preference heterogeneity in earnings. In words, the preference for savings must not increase "too quickly" across types, or else the second-order condition for earnings may be violated. The intuition for this result can be seen from the definition of the potentially optimal LED system. If the marginal rate of substitution for saving, $\mathcal{S}$, increases very quickly with income at some point in the allocation, then the savings tax rate $\tau_s(z)$ must rise very quickly with z at that point, by Equation (37). Since the savings tax rate $\tau_s(z)$ applies to total savings (including inframarginal savings), this increase in $\tau_s(z)$ must be offset by a sharp decrease in $T_z(z)$ at the same point in the distribution, by Equation (38). Yet a sufficiently steep decrease in $T_z(z)$ will cause the second-order condition for earnings choice—holding fixed savings choice—to be violated.

Step 3: Global maxima. Having presented conditions under which the bundle $(c^*(\theta), s^*(\theta), z^*(\theta))$ assigned to type θ is a local optimum under the candidate SN and LED tax systems, we now present a set of regularity conditions ensuring that these local optima are also global optima.

Proposition 10. *Assume single crossing conditions for earnings and savings ($\mathcal{Z}'_\theta \geq 0$ and $\mathcal{S}'_\theta \geq 0$), that preferences are weakly separable ($U''_{cz} = 0$ and $U''_{sz} = 0$), and that commodities c and s are weak complements ($U''_{cs} \geq 0$). If $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$ constitutes a set of local optima for types θ under a smooth tax system $\mathcal{T}$, local optima correspond to global optima when*

1. *$\mathcal{T}$ is a SN system, and we have that for all $s > s^*(\theta)$ and θ, $\frac{-U''_{ss}(c(s,\theta),s,z^*(\theta);\theta)}{U'_s(c(s,\theta),s,z^*(\theta);\theta)} > \frac{-T''_{ss}(s)}{1+T'_s(s)}$.*
2. *$\mathcal{T}$ is a LED system, and we have that*
   (a) for all $s < s^*(\theta)$ and θ, $\frac{-U''_{cc}(c(s,\theta),s,z^*(\theta);\theta)}{U'_c(c(s,\theta),s,z^*(\theta);\theta)} > \frac{1}{1+\tau_s(z^*(\theta))}\ \frac{\tau'_s(z^*(\theta))}{1-\tau'_s(z^*(\theta))s-T'_z(z^*(\theta))}$,
   (b) for all $s > s^*(\theta)$ and θ, $\frac{-U''_{ss}(c(s,\theta),s,z^*(\theta);\theta)}{U'_s(c(s,\theta),s,z^*(\theta);\theta)} > \frac{\tau'_s(z^*(\theta))}{1-\tau'_s(z^*(\theta))s-T'_z(z^*(\theta))}$.

where $c(s,\theta) := z^*(\theta) - s - \mathcal{T}(s, z^*(\theta))$

In essence, global optimality is ensured under the following assumptions. First, higher types θ derive higher gains from working and allocating those gains to consumption or savings — generalized single-crossing conditions. Second, additive separability of consumption and savings from labor. Third, the utility function U is sufficiently concave in consumption and savings. For the case of *SN* tax systems, condition 1 imposes a particular concavity requirement with respect to savings. For the case of LED tax systems, condition 2 imposes particular concavity requirements with respect to both consumption and savings. Notably, these concavity conditions need only be checked when earnings are fixed at each type's assigned earnings level $z^*(\theta)$.

We can naturally apply this result to the candidate SN tax system defined in Equations (35) and (36), and to the candidate LED tax system defined in Equations (37) and (38). Because these candidate tax systems are defined in terms of agents' preferences and optimal allocations, we can then express conditions 1 and 2 fully in terms of agents' preferences and optimal allocations. Yet, these general formulas are somewhat involved and we only check that they hold for particular utility functions.

B Proofs

B.1 Proof of Proposition 1 (implementability with a smooth tax system)

We provide a constructive proof of Proposition 1: we construct a smooth tax system that implements the optimal incentive-compatible allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$ by introducing penalties for deviations away from these allocations. This proof relies on Lemma 2 and Lemma 3 that we derive at the end of this subsection.

By Assumption 2, functions $s^*(\theta)$ and $z^*(\theta)$ are smooth and strictly increasing in θ over the compact set of types $[\underline{\theta}\,;\,\overline{\theta}]$ such that there exists a mapping $s^*(z)$ which denotes the savings level associated with earnings level z at the optimal incentive-compatible allocation. Let $\bar{z} := z^*\left(\overline{\theta}\right)$ and $\underline{z} := z^*(\underline{\theta})$ denote the maximal and minimal, respectively, earnings levels in the allocation. Let $\overline{s} := s^*(\overline{z})$ and $\underline{s} := s^*(\underline{z})$ denote the corresponding maximal and minimal savings levels.

Step 1: Defining the smooth tax system. Intuitively, we start from a separable and smooth tax system $T_s(s) + T_z(z)$ that satisfies type-specific feasibility and agents' first-order conditions at the optimal incentive-compatible allocation. We then add quadratic penalty terms parametrized by k for deviations from this allocation. For a given deviation, this allows to make the penalty arbitrarily large and enables us to make agents problem locally concave around the optimal incentive-compatible allocation. The other terms that we add are there to guarantee the smoothness of the penalized tax system $\mathcal{T}(s, z; k)$ at the boundaries of the set of optimal allocations.

Formally, $\mathcal{T}_k = \mathcal{T}(s, z; k)$ is defined by:

1. $T_s(\underline{s}) = 0$ and $T_z(\underline{z}) = z^*(\underline{\theta}) - c^*(\underline{\theta}) - s^*(\underline{\theta})$
2. $\forall z \in [\,\underline{z}\,;\,\overline{z}\,]\,, T_z'(z) = \mathcal{Z}(c^*(\theta_z), s^*(\theta_z), z^*(\theta_z); \theta_z) + 1$ with θ_z such that $z = z^*(\theta_z)$
3. $\forall s \in [\,\underline{s}\,;\,\overline{s}\,]\,, T_s'(s) = \mathcal{S}(c^*(\theta_s), s^*(\theta_s), z^*(\theta_s); \theta_s) - 1$ with θ_s such that $s = s^*(\theta_s)$

4. $$\mathcal{T}(s,z;k)=\begin{cases}T_s(s)+T_z(z)+k(s-s^*(z))^2 & \text{if } \underline{z}\leq z\leq\bar{z}, \underline{s}\leq s\leq\overline{s}\\ T_s(\underline{s})+T_z(z)+k(s-s^*(z))^2+T_s'(\underline{s})(s-\underline{s}) & \text{if } \underline{z}\leq z\leq\bar{z}, s<\underline{s}\\ T_s(\overline{s})+T_z(z)+k(s-s^*(z))^2+T_s'(\overline{s})(s-\overline{s}) & \text{if } \underline{z}\leq z\leq\bar{z}, s>\overline{s}\\ T_s(s)+T_z(\underline{z})+k(s-\underline{s})^2+k(z-\underline{z})^2+T_z'(\underline{z})(z-\underline{z}) & \text{if } z<\underline{z}, \underline{s}\leq s\leq\overline{s}\\ T_s(\underline{s})+T_z(\underline{z})+k(s-\underline{s})^2+k(z-\underline{z})^2+T_z'(\underline{z})(z-\underline{z})+T_s'(\underline{s})(s-\underline{s}) & \text{if } z<\underline{z}, s<\underline{s}\\ T_s(\overline{s})+T_z(\underline{z})+k(s-\underline{s})^2+k(z-\underline{z})^2+T_z'(\underline{z})(z-\underline{z})+T_s'(\overline{s})(s-\overline{s}) & \text{if } z<\underline{z}, s>\overline{s}\\ T_s(s)+T_z(\bar{z})+k(s-\overline{s})^2+k(z-\bar{z})^2+T_z'(\bar{z})(z-\bar{z}) & \text{if } z>\bar{z}, \underline{s}\leq s\leq\overline{s}\\ T_s(\overline{s})+T_z(\bar{z})+k(s-\overline{s})^2+k(z-\bar{z})^2+T_z'(\bar{z})(z-\bar{z})+T_s'(\overline{s})(s-\overline{s}) & \text{if } z>\bar{z}, s>\overline{s}\\ T_s(\underline{s})+T_z(\bar{z})+k(s-\overline{s})^2+k(z-\bar{z})^2+T_z'(\bar{z})(z-\bar{z})+T_s'(\underline{s})(s-\underline{s}) & \text{if } z>\bar{z}, s<\underline{s}\end{cases}$$

Assumptions 1 and 2 guarantee that the separable tax system $T_s(s)+T_z(z)$ is smooth i.e. a twice continuously differentiable function. Our construction of the penalized tax system $\mathcal{T}_k=\mathcal{T}(s,z;k)$ guarantees that it inherits this smoothness property.

Step 2: Local maxima for sufficiently large k. For a given agent θ, we show that the bundle $(c^*(\theta), s^*(\theta), z^*(\theta))$ is a local optimum under the tax system $\mathcal{T}_k=\mathcal{T}(s,z;k)$ for sufficiently large k. To do so, we first establish that type-specific feasibility is satisfied together with the first-order conditions of agent θ maximization problem. We then show that for sufficiently large k, second-order conditions are also satisfied implying that the intended bundle is a local maximum.

The previous definition of the tax system implies

$$\mathcal{T}_z'(s^*(\theta),z^*(\theta);k)=T_z'(z^*(\theta))=\mathcal{Z}(c^*(\theta),s^*(\theta),z^*(\theta);\theta)+1$$
$$\mathcal{T}_s'(s^*(\theta),z^*(\theta);k)=T_s'(s^*(\theta))=\mathcal{S}(c^*(\theta),s^*(\theta),z^*(\theta);\theta)-1$$

meaning type-specific feasibility is satisfied by Lemma 2 (see below).

Now, defining

$$V(s,z;\theta,k):=U(z-s-\mathcal{T}(s,z;k),s,z;\theta) \tag{42}$$

first-order conditions of agent θ choice of savings s and earnings z are

$$V_s'(s,z;\theta,k)=-(1+\mathcal{T}_s'(s,z;k))U_c'(z-s-\mathcal{T}(s,z;k),s,z;\theta)+U_s'(z-s-\mathcal{T}(s,z;k),s,z;\theta)=0$$
$$V_z'(s,z;\theta,k)=(1-\mathcal{T}_z'(s,z;k))U_c'(z-s-\mathcal{T}(s,z;k),s,z;\theta)+U_z'(z-s-\mathcal{T}(s,z;k),s,z;\theta)=0$$

and they are by construction satisfied at $(s^*(\theta),z^*(\theta))$ for each type θ.

Using Lemma 3 (see below), second-order conditions at $(s^*(\theta), z^*(\theta))$ are

$$V''_{ss} = \frac{U'_z}{s^{*\prime}(z^*)}\mathcal{S}'_c - \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{S}'_z - \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta + \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{T}''_{sz} \leq 0 \tag{43}$$

$$V''_{zz} = U'_s s^{*\prime}(z^*)\mathcal{Z}'_c - U'_c s^{*\prime}(z^*)\mathcal{Z}'_s - \frac{U'_c}{z^{*\prime}(\theta)}\mathcal{Z}'_\theta + U'_c s^{*\prime}(z^*)\mathcal{T}''_{sz} \leq 0 \tag{44}$$

$$\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \frac{U'_c}{s^{*\prime}(\theta)}\Bigg[\left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\mathcal{Z}'_\theta + \left(U'_s\mathcal{Z}'_c - U'_c\mathcal{Z}'_s - U'_c\frac{\mathcal{Z}'_\theta}{s^{*\prime}(\theta)}\right)s^{*\prime}(z^*)\mathcal{S}'_\theta$$
$$+ \left(\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta\right)U'_c\mathcal{T}''_{sz}\Bigg] \leq 0 \tag{45}$$

where U, $\mathcal{S}$, and $\mathcal{Z}$ are smooth functions implying that their derivatives are continuous functions over compact spaces, and are thus bounded. Now, by definition of $\mathcal{T}_k = \mathcal{T}(s, z; k)$ we have $\mathcal{T}''_{sz} = -2ks^{*\prime}(z)$ which is negative for any $k \geq 0$ and increasing in magnitude with k.

Noting $U'_c \geq 0$ and $s^{*\prime}(z) \geq 0$, it implies that V''_{ss} and V''_{zz} must be negative for sufficiently large k thanks to the last term since other terms are bounded and do not depend on k. By the same reasoning, under the extended Spence-Mirrlees single-crossing assumption that $\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta \geq 0$, we also have that $\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz}$ must be negative for sufficiently large k.

This shows that for a given agent θ, there exists a k_0 such that for all $k \geq k_0$ the allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$ is a local optimum to agent θ maximization problem under the smooth penalized tax system $\mathcal{T}_k = \mathcal{T}(s, z; k)$.

Step 3: Global maxima for sufficiently large k. Let $s_{\mathcal{T}_k}(\theta)$ and $z_{\mathcal{T}_k}(\theta)$ denote the level of savings and earnings, respectively, that a type θ chooses given a smooth penalized tax system $\mathcal{T}_k$. To prove implementability of optimal incentive-compatible allocations, we show that there exists a sufficiently large k such that for all θ, $s_{\mathcal{T}_k}(\theta) = s^*(\theta)$ and $z_{\mathcal{T}_k}(\theta) = z^*(\theta)$.

Let's proceed by contradiction, and suppose that it is not the case. Then, there exists an infinite sequence of types θ_k, choosing savings $s_{\mathcal{T}_k}(\theta_k) \neq s^*(\theta_k)$ and earnings $z_{\mathcal{T}_k}(\theta_k) \neq z^*(\theta_k)$ under tax system $\mathcal{T}_k$, and enjoying utility gains from this "deviation" to allocation $(s_{\mathcal{T}_k}(\theta_k), z_{\mathcal{T}_k}(\theta_k))$.

First, the fact that we impose quadratic penalties for earnings choices outside of $[\underline{z}; \overline{z}]$ guarantees that for any $\varepsilon > 0$, there exists k_1, such that for all $k \geq k_1$ and for all types θ, agents' earnings choices belong to the compact set $[\underline{z} - \varepsilon; \overline{z} + \varepsilon]$. Indeed, starting from a given earnings level $z > \overline{z} + \varepsilon$, the utility change associated with an earnings change is $[(1 - \mathcal{T}'_z)U'_c + U'_z]\,dz$. By construction, the marginal earnings tax rate in this region is $\mathcal{T}'_z = 2k(z - \bar{z}) + T'_z(\bar{z})$. Noting that $U'_c > 0$, $U'_z < 0$, and that the type space is compact, we can make for all individuals the utility change from a decrease in earnings arbitrarily positive for sufficiently large k. This shows that all individuals choose earnings $z \leq \overline{z} + \varepsilon$ for sufficiently large k. Symetrically, we can show that all individuals choose earnings $z \geq \underline{z} - \varepsilon$ for sufficiently large k.

Second, since earnings shape agents' disposable income, the fact that earnings belong to a compact set for sufficiently large k implies that agents' allocation choices also belong to a compact set. Indeed, for sufficiently large k agents' allocation choices must belong to the set of (c, s, z) such that $c \geq 0$, $s \geq 0$, $z \in [\underline{z} - \varepsilon\,;\, \overline{z} + \varepsilon]$, and $c + s \leq z - \mathcal{T}(s, z; k)$ where the tax function is smooth and finite. These constraints make the space of allocations compact.

As a result, the infinite sequence $(\theta_k, s_{\mathcal{T}_k}(\theta_k), z_{\mathcal{T}_k}(\theta_k))$ belongs to a compact space of allocations and types, it is thus bounded. By the Bolzano–Weierstrass theorem, this means that there exists a convergent subsequence $\left(\theta_j, s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right) \to (\hat{\theta}, \hat{s}, \hat{z})$. Since all types θ_j belong to $\left[\underline{\theta}\,;\,\overline{\theta}\right]$, we know that the limit type $\hat{\theta}$ must belong to $\left[\underline{\theta}\,;\,\overline{\theta}\right]$. Now, from the previous paragraph, agents' earnings choices have to be arbitrarily close to $[\underline{z}\,;\,\overline{z}]$ as the penalty grows. This implies that the limit $\hat{z}$ must belong to $[\underline{z}\,;\,\overline{z}]$.

Next, we establish that the limit $\hat{s}$ must be such that $\hat{s} = s^*(\hat{z})$. Fix an earnings level $z \in [\underline{z}\,;\,\overline{z}]$, starting from a savings level $s \neq s^*(z)$ the utility change associated with a savings change is $[-(1 + \mathcal{T}_s')\, U_c' + U_s']\, ds$. Assuming without loss of generality that s belongs to $[\underline{s}\,;\,\overline{s}]$, the marginal savings tax rate in this region is $\mathcal{T}_s' = T_s'(s) + 2k(s - s^*(z))$. Noting that $U_c' > 0$, and that U_s' is bounded, we can make the utility gains from a savings change towards $s^*(z)$ arbitrarily large for sufficiently large k. As a result, for any $\varepsilon > 0$, there exists k_2 such that for all $k \geq k_2$, agent $\hat{\theta}$ chooses savings $s \in [s^*(z) - \varepsilon\,;\, s^*(z) + \varepsilon]$ for a fixed z.[15] Since agent $\hat{\theta}$ savings choice can be made arbitrarily close to $s^*(z)$ for sufficiently large k, we must have at the limit $s = s^*(z)$. Now, because earnings z converge to $\hat{z}$ and the function $s^*(z)$ is by assumption continuous, we must have at the limit $\hat{s} = s^*(\hat{z})$.

We have thus established that the limit $(\hat{\theta}, \hat{s}, \hat{z})$ is such that $\hat{\theta} \in \left[\underline{\theta}\,;\,\overline{\theta}\right]$, $\hat{z} \in [\underline{z}\,;\,\overline{z}]$, and $\hat{s} = s^*(\hat{z})$. This means that the limit allocation $(\hat{c}, \hat{s}, \hat{z})$ belongs to the set of optimal incentive-compatible allocations. Given our continuity and monotonicity assumptions, this implies that it is the optimal allocation of some type θ and it has to be by definition that of agent $\hat{\theta}$. Hence, $(\hat{c}, \hat{s}, \hat{z}) = (c^*(\hat{\theta}), s^*(\hat{\theta}), z^*(\hat{\theta}))$.

To complete the proof and find a contradiction, fix a value $k^\dagger$ that is large enough such that second-order conditions hold for type $\hat{\theta}$ at allocation $(s^*(\hat{\theta}), z^*(\hat{\theta}))$ under tax system $\mathcal{T}_{k^\dagger}$ – this $k^\dagger$ exists by step 2. This implies that there exists an open set N containing $(s^*(\hat{\theta}), z^*(\hat{\theta}))$ such that $V(s, z; \hat{\theta}, k^\dagger)$ is strictly concave over $(s, z) \in N$.

Now, consider types θ^j with $j \geq k^\dagger$. Since these agents belong to the previously defined subsequence, they prefer allocation $\left(s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right)$ to allocation $(s^*(\theta_j), z^*(\theta_j))$ under tax system $\mathcal{T}_j$. Because penalties are increasingly large and $j \geq k^\dagger$, this implies that agents θ^j also prefer allocation $\left(s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right)$ to allocation $(s^*(\theta_j), z^*(\theta_j))$ under tax system $\mathcal{T}_{k^\dagger}$.

Yet, by continuity, the function $V\left(s, z; \theta_j, k^\dagger\right)$ gets arbitrarily close to the function $V(s, z; \hat{\theta}, k^\dagger)$ for sufficiently large j since $\theta_j \to \hat{\theta}$. For the same reason, $(s^*(\theta_j), z^*(\theta_j)) \to (s^*(\hat{\theta}), z^*(\hat{\theta}))$. Moreover, by definition $\left(s_{\mathcal{T}_j}(\theta_j), z_{\mathcal{T}_j}(\theta_j)\right) \to (\hat{s}, \hat{z})$. As a result, for any open set $N' \subsetneq N$ containing $(s^*(\hat{\theta}), z^*(\hat{\theta}))$, there exists a $j^\dagger \geq k^\dagger$ such that $V(s, z; \theta_{j^\dagger}, k^\dagger)$ is strictly concave over $(s, z) \in N'$ and such that both $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$ and $\left(s_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger}), z_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger})\right)$ belong to the set N'.

[15] A way to see this is that the marginal rate of substitution between consumption and savings $\mathcal{S}$ is continuous on a compact space and thus bounded, whereas the marginal tax rate which is parametrized by k can be made arbitrarily large. As a result, agents' first-order condition can never hold for sufficiently large k, while we can similarly exclude corner solutions for sufficiently large k.

Since $V(s, z; \theta_{j^\dagger}, k^\dagger)$ is strictly concave over $(s, z) \in N'$ it has a unique optimum on N'. By definition of $\mathcal{T}_{k^\dagger}$, agent $\theta_{j^\dagger}$ first-order conditions are satisfied at $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$. Hence, $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$ is agent $\theta_{j^\dagger}$ maximum under the tax system $\mathcal{T}_{k^\dagger}$. This contradicts the fact established above that agent $\theta_{j^\dagger}$ prefers $\left(s_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger}), z_{\mathcal{T}_{j^\dagger}}(\theta_{j^\dagger})\right)$ to allocation $\left(s^*(\theta_{j^\dagger}), z^*(\theta_{j^\dagger})\right)$ under tax system $\mathcal{T}_{k^\dagger}$, which completes the proof.

Lemma for type-specific feasibility.

Lemma 2. *A smooth tax system $\mathcal{T}$ satisfies type-specific feasibility over the compact type space $\left[\underline{\theta}\,;\overline{\theta}\right]$ if it satisfies the following conditions*

1. $\mathcal{T}(s^*(\underline{\theta}), z^*(\underline{\theta})) = z^*(\underline{\theta}) - c^*(\underline{\theta}) - s^*(\underline{\theta})$
2. $\mathcal{T}_z'(s^*(\theta), z^*(\theta)) = \mathcal{Z}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) + 1$
3. $\mathcal{T}_s'(s^*(\theta), z^*(\theta)) = \mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta); \theta) - 1$

Proof. Consider the type-specific feasible tax system $T^*_\theta(\theta) = z^*(\theta) - s^*(\theta) - c^*(\theta)$, and note that the lemma amounts to showing that $T^*_\theta(\theta) = \mathcal{T}(s^*(\theta), z^*(\theta))$ for all θ. To that end, note that the first-order condition for truthful reporting of θ under the optimal mechanism implies

$$U_c' \cdot (z'(\theta) - s'(\theta) - T^{*\prime}(\theta)) + U_s' \cdot s'(\theta) + U_z' \cdot z'(\theta) = 0,$$

with derivatives evaluated at the optimal allocation. This can be rearranged as

$$\begin{aligned} T^{*\prime}(\theta) &= \left(\frac{U_s'}{U_c'} - 1\right) s'(\theta) + \left(\frac{U_z'}{U_c'} + 1\right) z'(\theta) \\ &= \mathcal{T}_s'(s^*(\theta)) s^{*\prime}(\theta) + \mathcal{T}_z'(z^*(\theta)) z^{*\prime}(\theta). \end{aligned}$$

It thus follows that

$$\begin{aligned} \mathcal{T}(s^*(\theta), z^*(\theta)) - \mathcal{T}(s^*(\underline{\theta}), z^*(\underline{\theta})) &= \int_{\vartheta=\underline{\theta}}^{\vartheta=\theta} \left(\mathcal{T}_s'(s^*(\vartheta)) s^{*\prime}(\vartheta) + \mathcal{T}_z'(z^*(\vartheta)) z^{*\prime}(\vartheta)\right) d\vartheta \\ &= T^*_\theta(\theta) - T^*_\theta(\underline{\theta}) \end{aligned}$$

Since $\mathcal{T}(s^*(\underline{\theta}), z^*(\underline{\theta})) = T^*_\theta(\underline{\theta})$, this implies that $\mathcal{T}(s^*(\theta), z^*(\theta)) = T^*_\theta(\theta)$ for all θ. The smooth tax system $\mathcal{T}$ therefore satisfies type-specific feasibility. □

Lemma on second-order conditions.

Lemma 3. *Consider a smooth tax system $\mathcal{T}$ satisfying the conditions in Lemma 2 and define*

$$V(s, z; \theta) := U(z - s - \mathcal{T}(s, z), s, z; \theta). \tag{46}$$

When evaluated at allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$*, we show that*

$$V''_{ss} = \frac{U'_z}{s^{*\prime}(z^*)}\mathcal{S}'_c - \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{S}'_z - \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta + \frac{U'_c}{s^{*\prime}(z^*)}\mathcal{T}''_{sz} \tag{47}$$

$$V''_{zz} = U'_s s^{*\prime}(z^*)\mathcal{Z}'_c - U'_c s^{*\prime}(z^*)\mathcal{Z}'_s - \frac{U'_c}{z^{*\prime}(\theta)}\mathcal{Z}'_\theta + U'_c s^{*\prime}(z^*)\mathcal{T}''_{sz} \tag{48}$$

$$\begin{aligned}\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \frac{U'_c}{s^{*\prime}(\theta)}\Bigg[&\left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\mathcal{Z}'_\theta + \left(U'_s\mathcal{Z}'_c - U'_c\mathcal{Z}'_s - U'_c\frac{\mathcal{Z}'_\theta}{s^{*\prime}(\theta)}\right)s^{*\prime}(z^*)\mathcal{S}'_\theta \\ &+ \left(\mathcal{Z}'_\theta + s^{*\prime}(z^*)\mathcal{S}'_\theta\right)U'_c\mathcal{T}''_{sz}\Bigg]\end{aligned} \tag{49}$$

where we denote $s^{*\prime}(z^*) := \frac{s^{*\prime}(\theta)}{z^{*\prime}(\theta)}$.

Proof. First-order derivatives are

$$\begin{aligned}V'_s(s,z;\theta) &= -(1+\mathcal{T}'_s(s,z))U'_c(z-s-\mathcal{T}(s,z),s,z;\theta) + U'_s(z-s-\mathcal{T}(s,z),s,z;\theta) \\ V'_z(s,z;\theta) &= (1-\mathcal{T}'_z(s,z))U'_c(z-s-\mathcal{T}(s,z),s,z;\theta) + U'_z(z-s-\mathcal{T}(s,z),s,z;\theta)\end{aligned}$$

Second-order derivatives are

$$V''_{ss}(s,z;\theta) = -\mathcal{T}''_{ss}U'_c - (1+\mathcal{T}'_s)\left(-(1+\mathcal{T}'_s)U''_{cc} + U''_{cs}\right) - (1+\mathcal{T}'_s)U''_{cs} + U''_{ss} \tag{50}$$

$$V''_{zz}(s,z;\theta) = -\mathcal{T}''_{zz}U'_c + (1-\mathcal{T}'_z)\left((1-\mathcal{T}'_z)U''_{cc} + U''_{cz}\right) + (1-\mathcal{T}'_z)U''_{cz} + U''_{zz} \tag{51}$$

To obtain the first result of the Lemma, we compute $\mathcal{T}''_{ss}$ by differentiating both sides of $\mathcal{T}'_s(s^*(\theta), z^*(\theta)) = \mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta);\theta) - 1$ with respect to θ

$$\begin{aligned}\mathcal{T}''_{ss}s^{*\prime}(\theta) + \mathcal{T}''_{sz}z^{*\prime}(\theta) &= \frac{d}{d\theta}\mathcal{S}(c^*(\theta), s^*(\theta), z^*(\theta);\theta) \\ &= \mathcal{S}'_c c^{*\prime}(\theta) + \mathcal{S}'_s s^{*\prime}(\theta) + \mathcal{S}'_z z^{*\prime}(\theta) + \mathcal{S}'_\theta\end{aligned}$$

plugging in $c^{*\prime}(\theta) = (1-\mathcal{T}'_z)\, z^{*\prime}(\theta) - (1+\mathcal{T}'_s)\, s^{*\prime}(\theta)$ and denoting $s^{*\prime}(z^*) := s^{*\prime}(\theta)/z^{*\prime}(\theta)$, the previous expression can be rearranged as

$$\mathcal{T}''_{ss} = \mathcal{S}'_c\frac{1-\mathcal{T}'_z}{s^{*\prime}(z^*)} - \mathcal{S}'_c(1+\mathcal{T}'_s) + \mathcal{S}'_s + \frac{\mathcal{S}'_z}{s^{*\prime}(z^*)} + \frac{\mathcal{S}'_\theta}{s^{*\prime}(\theta)} - \frac{\mathcal{T}''_{sz}}{s^{*\prime}(z^*)}. \tag{52}$$

Moreover, we differentiate Equation (2) to express the derivative of $\mathcal{S}$ with respect to c as

$$\begin{aligned}\mathcal{S}'_c(c^*(\theta), s^*(\theta), z^*(\theta);\theta) &= \frac{U'_cU''_{sc} - U'_sU''_{cc}}{(U'_c)^2} \\ &= \frac{1}{U'_c}\left(-\frac{U'_s}{U'_c}U''_{cc} + U''_{sc}\right) \\ &= \frac{1}{U'_c}\left(-(1+\mathcal{T}'_s)U''_{cc} + U''_{sc}\right)\end{aligned} \tag{53}$$

and the derivative of $\mathcal{S}$ with respect to s as

$$\begin{aligned}\mathcal{S}_s'(c^*(\theta), s^*(\theta), z^*(\theta); \theta) &= \frac{U_c' U_{ss}'' - U_s' U_{cs}''}{(U_c')^2} \\ &= \frac{1}{U_c'}\left(-\frac{U_s'}{U_c'} U_{cs}'' + U_{ss}''\right) \\ &= \frac{1}{U_c'}\left(-(1+\mathcal{T}_s') U_{cs}'' + U_{ss}''\right). \end{aligned} \tag{54}$$

Substituting equations (52), (53) and (54) into (50), we have

$$\begin{aligned} V_{ss}''\left(s^*\left(\theta\right), z^*\left(\theta\right); \theta\right) &= -U_c' \cdot \left(\mathcal{S}_c' \frac{1-\mathcal{T}_z'}{s^{*\prime}(z)} - \mathcal{S}_c'(1+\mathcal{T}_s') + \mathcal{S}_s' + \frac{\mathcal{S}_z'}{s^{*\prime}(z)} + \frac{\mathcal{S}_\theta'}{s^{*\prime}(\theta)} - \frac{\mathcal{T}_{sz}''}{s^{*\prime}(z)}\right) - (1+\mathcal{T}_s') U_s' \mathcal{S}_c' + U_c' \mathcal{S}_s' \\ &= -U_c' \cdot \left(\frac{1-T_z'}{s^{*\prime}(z)} \mathcal{S}_c' + \frac{1}{s^{*\prime}(z)} \mathcal{S}_z' + \frac{1}{s^{*\prime}(\theta)} \mathcal{S}_\theta' - \frac{\mathcal{T}_{sz}''}{s^{*\prime}(z)}\right) \\ &= \frac{U_z'}{s^{*\prime}(z)} \mathcal{S}_c' - \frac{U_c'}{s^{*\prime}(z^*)} \mathcal{S}_z' - \frac{U_c'}{s^{*\prime}(\theta)} \mathcal{S}_\theta' + \frac{U_c'}{s^{*\prime}(z^*)} \mathcal{T}_{sz}''. \end{aligned} \tag{55}$$

where we have used $U_z' = -U_c'\left(1-\mathcal{T}_z'\right)$ in the last line.

Similarly, we can obtain the second result of the Lemma by writing $\mathcal{T}_{zz}''$ as

$$\mathcal{T}_{zz}'' = \mathcal{Z}_c'\left(1-\mathcal{T}_z'\right) - \mathcal{Z}_c'\left(1+\mathcal{T}_s'\right) s^{*\prime}(z^*) + \mathcal{Z}_s' s^{*\prime}(z^*) + \mathcal{Z}_z' + \frac{\mathcal{Z}_\theta'}{z^{*\prime}(\theta)} - \mathcal{T}_{sz}'' s^{*\prime}(z^*). \tag{56}$$

Using

$$\mathcal{Z}_c' = \frac{1}{U_c'}\left(U_{cz}'' + \left(1-\mathcal{T}_z'\right) U_{cc}''\right)$$

as well as

$$\mathcal{Z}_z' = \frac{1}{U_c'}\left(U_{zz}'' + \left(1-\mathcal{T}_z'\right) U_{cz}''\right)$$

we get

$$V_{zz}''\left(s^*\left(\theta\right), z^*\left(\theta\right); \theta\right) = U_s' s^{*\prime}(z^*) \mathcal{Z}_c' - U_c' s^{*\prime}(z^*) \mathcal{Z}_s' - U_c' \frac{\mathcal{Z}_\theta'}{z^{*\prime}(\theta)} + U_c' \mathcal{T}_{sz}'' s^{*\prime}(z^*). \tag{57}$$

Finally, to obtain the third result of the Lemma, we must compute $\left(V_{sz}''\right)^2 - V_{ss}'' V_{zz}''$. Note that the first-order condition $V_s'(s^*(\theta), z^*(\theta); \theta) = 0$ holds at every θ by construction. Differentiating with respect to θ we get

$$\frac{d}{d\theta} V_s'(s^*(\theta), z^*(\theta); \theta) = V_{ss}'' s^{*\prime}(\theta) + V_{sz}'' z^{*\prime}(\theta) + V_{s\theta}'' = 0 \tag{58}$$

which we can rearrange as

$$-V_{sz}'' = V_{ss}'' s^{*\prime}(z^*) + \frac{V_{s\theta}''}{z^{*\prime}(\theta)}. \tag{59}$$

Similarly, by totally differentiating the first-order condition $V_z'(s^*(\theta), z^*(\theta); \theta) = 0$ and rearranging we find

$$-V_{sz}'' = \frac{V_{zz}''}{s^{*\prime}(z^*)} + \frac{V_{z\theta}''}{s^{*\prime}(\theta)}. \tag{60}$$

Writing $(V_{sz}'')^2$ as the product of the right-hand sides of Equations (59) and (60) yields

$$\begin{aligned}(V_{sz}'')^2 =& \left(V_{ss}''s^{*\prime}(z) + \frac{V_{s\theta}''}{z^{*\prime}(\theta)}\right)\left(\frac{V_{zz}''}{s^{*\prime}(z)} + \frac{V_{z\theta}''}{s^{*\prime}(\theta)}\right)\\ =& V_{ss}''V_{zz}'' + \frac{1}{z^{*\prime}(\theta)}V_{ss}''V_{z\theta}'' + \frac{1}{s^{*\prime}(\theta)}V_{zz}''V_{s\theta}'' + \frac{1}{s^{*\prime}(\theta)z^{*\prime}(\theta)}V_{s\theta}''V_{z\theta}'' \end{aligned} \tag{61}$$

Now from the definition $V(s,z;\theta) := U(z - s - \mathcal{T}(s,z), s, z; \theta)$, we can compute

$$\begin{aligned} V_{s\theta}''(s,z;\theta) &= -\left(1 + \mathcal{T}_s'(s,z)\right)U_{c\theta}'' + U_{s\theta}''\\ V_{z\theta}''(s,z;\theta) &= \left(1 - \mathcal{T}_z'(s,z)\right)U_{c\theta}'' + U_{z\theta}'' \end{aligned}$$

and use the fact that at allocation $(c^*(\theta), s^*(\theta), z^*(\theta))$ we have

$$\begin{aligned} \mathcal{S}_\theta' &= \frac{1}{U_c'}\left(U_{s\theta}'' - \left(1 + \mathcal{T}_s'\right)U_{c\theta}''\right)\\ \mathcal{Z}_\theta' &= \frac{1}{U_c'}\left(U_{z\theta}'' + \left(1 - \mathcal{T}_z'\right)U_{c\theta}''\right) \end{aligned}$$

to obtain

$$V_{s\theta}''(s^*(\theta), z^*(\theta); \theta) = U_c'\mathcal{S}_\theta' \tag{62}$$

$$V_{z\theta}''(s^*(\theta), z^*(\theta); \theta) = U_c'\mathcal{Z}_\theta'. \tag{63}$$

Substituting these into Equation (61) and rearranging, we have

$$(V_{sz}'')^2 - V_{ss}''V_{zz}'' = \frac{1}{z^{*\prime}(\theta)}V_{ss}''U_c'\mathcal{Z}_\theta' + \frac{1}{s^{*\prime}(\theta)}V_{zz}''U_c'\mathcal{S}_\theta' + \frac{1}{s^{*\prime}(\theta)z^{*\prime}(\theta)}\left(U_c'\right)^2\mathcal{S}_\theta'\mathcal{Z}_\theta'. \tag{64}$$

Expanding V_{ss}'' from Equation (55), and V_{zz}'' from Equation (57) yields after simplification

$$\begin{aligned}(V_{sz}'')^2 - V_{ss}''V_{zz}'' = \frac{U_c'}{s^{*\prime}(\theta)}\Bigg[&\left(U_z'\mathcal{S}_c' - U_c'\mathcal{S}_z'\right)\mathcal{Z}_\theta' + \left(U_s'\mathcal{Z}_c' - U_c'\mathcal{Z}_s' - U_c'\frac{\mathcal{Z}_\theta'}{s^{*\prime}(\theta)}\right)s^{*\prime}(z^*)\mathcal{S}_\theta'\\ &+ \left(\mathcal{Z}_\theta' + s^{*\prime}(z^*)\mathcal{S}_\theta'\right)U_c'\mathcal{T}_{sz}''\Bigg],\end{aligned}$$

which gives the third result of the Lemma above. □

B.2 Proof of Proposition 2 (measurement of preference heterogeneity)

We here derive the different expressions of the sufficient statistic $s'_{inc}(z)$ can be measured empirically.

Case 1. If agents preferences are weakly separable between the utility of consumption $u(.)$ and the disutility to work $k(.)$, agent θ problem writes

$$\max_{c,s,z} \; u(c,s;\theta) - k(z/w(\theta)) \;\; s.t. \; c \leq z - s - \mathcal{T}(s,z)$$

meaning that conditional on earnings z, savings $s(z;\theta)$ is defined as the solution to

$$-\left(1+\mathcal{T}'_s(s,z)\right) u'_c\left(z-s-\mathcal{T}(s,z), s;\theta\right) + u'_s\left(z-s-\mathcal{T}(s,z), s;\theta\right) = 0.$$

Differentiating this equation with respect to savings s and earnings z yields

$$\frac{\partial s}{\partial z} = -\frac{\left[-\mathcal{T}''_{sz}u'_c - (1+\mathcal{T}'_s)(1-\mathcal{T}'_z)u''_{cc} + (1-\mathcal{T}'_z)u''_{cs}\right]}{\left[-\mathcal{T}''_{ss}u'_c + (1+\mathcal{T}'_s)^2 u''_{cc} - 2(1+\mathcal{T}'_s(s,z))u''_{cs} + u''_{ss}\right]}.$$

Differentiating this equation with respect to savings s and disposable income y yields

$$\frac{\partial s}{\partial y} = -\frac{\left[-(1+\mathcal{T}'_s)u''_{cc} + u''_{cs}\right]}{\left[-\mathcal{T}''_{ss}u'_c + (1+\mathcal{T}'_s)^2 u''_{cc} - 2(1+\mathcal{T}'_s(s,z))u''_{cs} + u''_{ss}\right]}.$$

Hence, if $\mathcal{T}''_{sz} = 0$, we get

$$s'_{inc}(z) := \frac{\partial s(z;\theta)}{\partial z} = \left(1-\mathcal{T}'_z\right)\frac{\partial s}{\partial y} = \left(1-\mathcal{T}'_z\right)\frac{\eta_{s|z}(\theta)}{1+\mathcal{T}'_s},$$

where the last equality follows from the definition of $\eta_{s|z}(\theta)$. The intuition behind this result is that with separable preferences, savings s depend on earnings z only through disposable income $y = z - s - \mathcal{T}(s,z)$.

Case 2. If agents wage rates w and hours h are observable, and earnings z are given by $z = w \cdot h$, we can infer s'_{inc} from changes in wages through

$$\frac{\partial s}{\partial w} = \frac{\partial s(w\cdot h;\theta)}{\partial w} = \frac{\partial s(z;\theta)}{\partial z}\left(1+\frac{\partial h}{\partial w}\right)$$

$$\iff \frac{\partial s(z;\theta)}{\partial z} = \frac{\frac{\partial s}{\partial w}}{1+\frac{\partial h}{\partial w}} = s\frac{\frac{w}{s}\frac{\partial s}{\partial w}}{w + h\frac{w}{h}\frac{\partial h}{\partial w}}$$

$$\iff s'_{inc}(z) = s(z)\frac{\xi^s_w(z)}{w(z) + h(z)\xi^h_w(z)}$$

where $\xi_w^s(z) \equiv \frac{w(z)}{s(z)}\frac{\partial s(z)}{\partial w(z)}$ is individuals' elasticity of savings with respect to their wage rate, and $\xi_w^h(z) \equiv \frac{w(z)}{h(z)}\frac{\partial h(z)}{\partial w(z)}$ is individuals' elasticity of hours with respect to their wage rate.

Case 3. Otherwise, if we can measure the elasticity of savings s and earnings z upon a compensated change in the marginal earnings tax rate $\mathcal{T}_z'$, respectively denoted $\chi_s^c := -\frac{1-\mathcal{T}_z'}{s}\frac{\partial s}{\partial \mathcal{T}_z'}$ and $\zeta_z^c := -\frac{1-\mathcal{T}_z'}{z}\frac{\partial z}{\partial \mathcal{T}_z'}$, we then have

$$\begin{aligned}\frac{\partial s}{\partial \mathcal{T}_z'} &= \frac{\partial s(z;\theta)}{\partial z}\frac{\partial z}{\mathcal{T}_z'}\\ \iff \left(-\frac{s}{1-\mathcal{T}_z'}\chi_s^c\right) &= s_{inc}'(z)\left(-\frac{z}{1-\mathcal{T}_z'}\zeta_z^c\right)\\ \iff s_{inc}'(z) &= \frac{s(z)}{z}\frac{\chi_s^c(z)}{\zeta_z^c(z)}.\end{aligned}$$

B.3 Proof of Lemma 1 (characterization of equivalent earnings tax reforms)

Let

$$V(\mathcal{T}(.,z),z;\theta) = \max_s\ U(z-s-\mathcal{T}(s,z),s,z;\theta)$$

be agent θ indirect utility function at earnings z.

Consider a savings tax reform $d\mathcal{T}_s(s)$ that consists in a small increase $d\tau_s$ in the marginal savings tax rate in a bandwidth of savings $[s^*, s^*+ds]$, with $d\tau_s$ much smaller than ds:

$$d\mathcal{T}_s(s) = \begin{cases} 0 & if\ s \leq s^* \\ d\tau_s(s-s^*) & if\ s \in [s^*, s^*+ds] \\ d\tau_s ds & if\ s \geq s^*+ds \end{cases}$$

The idea is to construct for each type θ a perturbation of the earnings tax $d\mathcal{T}_z^\theta(z)$ that induces the same earnings response as the initial perturbation $d\mathcal{T}_s(s)$.

Suppose we define this perturbation for each type θ such that at all earnings z,

$$V(\mathcal{T}(.,z)+d\mathcal{T}_s(.),z;\theta) = V(\mathcal{T}(.,z)+d\mathcal{T}_z^\theta(.),z;\theta)$$

then, by construction, the perturbation $d\mathcal{T}_z^\theta(.)$ induces the same earnings response dz as the initial perturbation $d\mathcal{T}_s(s)$.

Moreover, both tax reforms must induce the same utility change for agent θ. Applying the envelope theorem to agents θ for whom $s(z;\theta) \in [s^*, s^*+ds]$, yields

$$\begin{aligned}U_c' \cdot d\tau_s(s(z;\theta)-s^*) &= U_c' \cdot d\mathcal{T}_z^\theta(z)\\ \iff d\mathcal{T}_z^\theta(z) &= (s(z;\theta)-s^*)\,d\tau_s.\end{aligned}$$

As can be seen differentiating both sides by earnings z, this implies that a small increase $d\tau_s$ in the marginal savings tax rate induces the same earnings change as a small increase $s'_{inc}(z)\,d\tau_s$ in the marginal earnings tax rate, which proves Lemma 1.

Note that this proof does not rely on heterogeneity being unidimensional, meaning that this result will equally apply when we consider multidimensional heterogeneity.

B.4 Proof of Proposition 3 (optimality formulas for smooth tax systems)

B.4.1 Marginal earnings tax rates

Reform. We consider a small reform at earnings level $z^* = z(\theta^*)$ that consists in a small increase $d\tau_z$ of the marginal tax rate on earnings in a small bandwidth dz. Formally,

$$d\mathcal{T}(s,z) = \begin{cases} 0 & if\ z \le z^* \\ d\tau_z(z - z^*) & if\ z \in [z^*, z^* + dz] \\ d\tau_z dz & if\ z \ge z^* + dz \end{cases}$$

We characterize the impact of this reform on the government objective function

$$\mathcal{L} = \int_z [\alpha(z)\, U(c(z), s(z), z; \theta(z)) + \lambda(\mathcal{T}(s,z) - E)]\, dH_z(z)$$

with λ the marginal value of public funds. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects*:

$$\int_{z^*}^{\bar{z}} \Big(1 - \frac{\alpha(z)}{\lambda} U'_c\left(c(z), s(z), z; \theta(z)\right)\Big)\, d\tau_z dz\, dH_z(z)$$

- *behavioral effects from earnings changes*:[16]

$$- \mathcal{T}'_z\left(s\left(z^*\right), z^*\right) \frac{z^*}{1 - \mathcal{T}'_z\left(s\left(z^*\right), z^*\right)} \zeta^c_z(z^*) d\tau_z\, dz h_z(z^*) \\ - \int_{z^*}^{\bar{z}} \mathcal{T}'_z\left(s\left(z\right), z\right) \frac{\eta_z(z)}{1 - \mathcal{T}'_z\left(s\left(z\right), z\right)} d\tau_z dz\, dH_z(z)$$

[16]Note that by definition elasticity concepts include all circularities and adjustments induced by tax reforms such that savings and earnings changes are simply given by

$$\begin{cases} dz = -\frac{z}{1-\mathcal{T}'_z}\zeta^c_z(z) d\mathcal{T}'_z(s,z) - \frac{\eta_z(z)}{1-\mathcal{T}'_z} d\mathcal{T}(s,z) \\ ds = -\frac{\eta_{s|z}(z)}{1+\mathcal{T}'_s} d\mathcal{T}(s,z) + s'_{inc}(z) dz \end{cases}$$

- *behavioral effects from savings changes*:

$$-\mathcal{T}_s'\left(s(z^*),z^*\right)s_{inc}'(z^*)\left[\frac{z^*}{1-\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)}\zeta_z^c(z^*)d\tau_z\right]dzh_z(z^*)$$
$$-\int_{z^*}^{\bar{z}}\mathcal{T}_s'\left(s(z),z\right)\left[\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'\left(s(z),z\right)}+s_{inc}'(z)\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}\right]d\tau_z dz\,dH_z(z)$$

Summing over these different effects yields the total impact of the reform

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{dz}=\int_{z^*}^{\bar{z}}\left(1-\hat{g}(z)\right)\,d\tau_z dH_z(z)$$
$$-\left(\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)+s_{inc}'(z^*)\mathcal{T}_s'\left(s(z^*),z^*\right)\right)\frac{z^*}{1-\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)}\zeta_z^c(z^*)d\tau_z\,h_z(z^*) \tag{65}$$

where $\hat{g}(z)$ represent social marginal welfare weights augmented with income effects, that is

$$\hat{g}(z)=\frac{\alpha(z)}{\lambda}U_c'\left(c(z),s(z),z;\theta(z)\right)+\mathcal{T}_z'\left(s\left(z\right),z\right)\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}$$
$$+\mathcal{T}_s'\left(s(z),z\right)\left(\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'\left(s(z),z\right)}+s_{inc}'(z)\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}\right)$$

Optimality. A direct implication of this result is a sufficient statistics characterization of the optimal marginal earnings tax rate. Indeed, at the optimum the reform should have a zero impact on the government objective, meaning that at each earnings z^* the optimal marginal earnings tax rate satisfies

$$\frac{\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)}{1-\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)}=\frac{1}{\zeta_z^c(z^*)}\frac{1}{z^*h_z(z^*)}\int_{z=z^*}^{\bar{z}}\left(1-\hat{g}(z)\right)dH_z(z)-s_{inc}'(z^*)\frac{\mathcal{T}_s'\left(s(z^*),z^*\right)}{1-\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)}$$

B.4.2 Marginal savings tax rates

Reform. We consider a small reform of the savings tax at savings level $s^*=s(\theta^*)$ that consists in a small increase $d\tau_s$ of the marginal tax rate on savings in a small bandwidth ds. Formally,

$$d\mathcal{T}(s,z)=\begin{cases}0 & if\ s\leq s^*\\ d\tau_s(s-s^*) & if\ s\in[s^*,s^*+ds]\\ d\tau_s ds & if\ s\geq s^*+ds\end{cases}$$

Assuming agents' preferences are smooth across the type distribution, there exists an increasing mapping between earnings z and savings s. Denote z^* the earnings level such that $s^*=s(z^*)$. We characterize the impact of this reform on the government objective function. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects:*

$$\int_{z\geq z^*}\left(1-\frac{\alpha(z)}{\lambda}U_c'\left(c(z),s(z),z;\theta(z)\right)\right)d\tau_s ds\, dH_z(z)$$

- *behavioral effects from earnings changes*:[17]

$$-\mathcal{T}_z'(s^*,z^*)\left[\frac{z^*}{1-T_z'(z^*)}\zeta_z^c(z^*)\, s_{inc}'(z)d\tau_s\right]ds\,\frac{h_z(z^*)}{s'(z^*)}-\int_{z^*}^{\bar{z}}\mathcal{T}_z'\left(z\right)\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(z\right)}d\tau_s ds\, dH_z(z)$$

- *behavioral effects from savings changes*:

$$-\mathcal{T}_s'\left(s^*,z^*\right)\left[\frac{s(z^*)}{1+T_s'\left(s^*,z^*\right)}\zeta_{s|z}^c(z^*)d\tau_s+s_{inc}'(z^*)\frac{z^*}{1-T_z'\left(s^*,z^*\right)}\zeta_z^c(z^*)\, s_{inc}'(z^*)d\tau_s\right]ds\,\frac{h_z(z^*)}{s'(z^*)}$$
$$-\int_{z^*}^{\bar{z}}\mathcal{T}_s'\left(s(z)\right)\left[\frac{\eta_{s|z}(z)}{1+T_s'\left(s^*,z^*\right)}+s_{inc}'(z)\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(s^*,z^*\right)}\right]d\tau_s ds\, dH_z(z)$$

Summing over these different effects yields the total impact of the reform

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{ds}=s'\left(z^*\right)\int_{z^*}^{\bar{z}}\left(1-\hat{g}(z)\right)d\tau_s\, dH_z(z)-\left\{\mathcal{T}_s'\left(s(z^*)\right)\frac{s(z^*)}{1+\mathcal{T}_s'\left(s^*,z^*\right)}\zeta_{s|z}^c(z^*)\right.$$
$$\left.+\left[\mathcal{T}_z'(s^*,z^*)+s_{inc}'(z^*)\mathcal{T}_s'(s^*,z^*)\right]\frac{z^*}{1-\mathcal{T}_z'\left(s^*,z^*\right)}\zeta_z^c(z^*)s_{inc}'(z^*)\right\}d\tau_s\, h_z(z^*) \tag{67}$$

Optimality. A direct implication of this result is a sufficient statistics characterization of the optimal nonlinear savings tax schedule $T_s\left(s\right)$, given an existing and potentially suboptimal nonlinear earnings tax $T_z\left(z\right)$. Indeed, at the optimum the reform should have a zero impact on the government objective, meaning that the optimal nonlinear savings tax schedule verifies at each savings $s^*=s\left(z^*\right)$

$$\frac{\mathcal{T}_s'(s^*,z^*)}{1+\mathcal{T}_s'(s^*,z^*)}\zeta_{s|z}^c(z^*)s(z^*)h_z(z^*)$$
$$=s'\left(z^*\right)\int_{z^*}^{\bar{z}}\left(1-\hat{g}(z)\right)dH_z(z)-s_{inc}'(z^*)\frac{\mathcal{T}_z'(s^*,z^*)+s_{inc}'(z^*)\mathcal{T}_s'(s^*,z^*)}{1-\mathcal{T}_z'(s^*,z^*)}\zeta_z^c(z^*)z^*h_z(z^*) \tag{68}$$

Note that plugging in the formula for optimal marginal earnings tax rates, this formula can be

[17] Applying Lemma 1, earnings and savings changes are here given by

$$\begin{cases} dz=-\frac{z}{1-T_z'}\zeta_z^c(z)\delta T_z^{\theta\prime}-\frac{\eta_z(z)}{1-T_z'}\delta T_z^{\theta} \\ ds=-\frac{s(z)}{1+T_s'}\zeta_{s|z}^c(z)\delta T_s'-\frac{\eta_{s|z}(z)}{1+T_s'}\delta T_s+s_{inc}'(z)dz \end{cases} \tag{66}$$

where the equivalent reform T_z^{θ} is a small increase in the marginal earnings tax by $s_{inc}'(z^*)d\tau_s$ for types $\theta\in[\theta^*,\theta^*+d\theta]$, and a lump sum increase in tax liability by $d\tau_s ds$ for types above. Moreover, the mass of individuals in the bandwitdh is $\delta s\ h_s(s(z^*))=\delta s\,\frac{h_z(z^*)}{s'(z^*)}$.

rearranged as

$$\frac{\mathcal{T}_s'(s^*,z^*)}{1+\mathcal{T}_s'(s^*,z^*)} = s_{pref}'(z^*)\frac{1}{\zeta_{s|z}^c(z^*)}\frac{1}{h_z(z^*)s^*}\int_{z^*}^{\bar{z}}(1-\hat{g}(z))\,dH_z(z)$$

This optimal savings tax formula can also be obtained as a direct result through a joint reform of earnings and savings taxes at income level z^*. Starting from existing tax schedules, consider a *decrease* in marginal earnings tax rates by $d\tau_z$ in the bandwitdh $[z^*, z^*+dz]$ combined with an *increase* in marginal savings tax rates by $d\tau_s$ in the corresponding bandwidth $[s(z^*), s(z^*+dz)]$. Imposing $d\tau_z = s_{inc}'(z^*)d\tau_s$, this joint reform is such that the behavioral effects from earnings changes exactly cancel out (Lemma 1). Moreover, lump-sum changes induced by the two reforms are given by $d\tau_z dz = d\tau_s s_{inc}'(z^*)dz$ and by $d\tau_s ds = d\tau_s s'(z^*)dz$, meaning that the net lump-sum change is proportional to $s_{pref}'(z^*) = s'(z^*) - s_{inc}'(z^*)$. The total impact of the reform is thus

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{dz} = s_{pref}'(z^*)\int_{z^*}^{\bar{z}}(1-\hat{g}(z))\,d\tau_s\,dH_z(z) - \mathcal{T}_s'\left(s^*,z^*\right)\frac{s(z^*)}{1+\mathcal{T}_s'\left(s(z^*)\right)}\zeta_{s|z}^c(z^*)d\tau_s\,h_z(z^*)$$

and characterizing the optimum as a situation in which this type of joint reform does not affect the government objective function yields the result.

B.5 Proof of Corollary 1 (Pareto-efficiency formula for smooth tax systems)

We can combine formulas for optimal marginal earnings and marginal savings tax rates to obtain a characterization of Pareto-efficiency as a corollary of the optimum. Indeed, leveraging the previous optimal formula for marginal savings tax rates written in terms of $s_{pref}'(z^*)$, and replacing the integral term by its value given from the optimal formula for marginal earnings tax rates yields

$$\frac{\mathcal{T}_s'(s^*,z^*)}{1+T_s'(s^*,z^*)}\zeta_{s|z}^c(z^*)s^* = s_{pref}'(z^*)\frac{\mathcal{T}_z'(z^*)+s_{inc}'(z^*)\mathcal{T}_s'(s^*,z^*)}{1-\mathcal{T}_z'(s^*,z^*)}\zeta_z^c(z^*)z^*.$$

This Pareto-efficiency formula can also be obtained as a direct result through a joint reform of income and savings tax at income level z^*. Starting from existing tax schedules, consider an *increase* in marginal earnings tax rates by $d\tau_z$ in the bandwitdh $[z^*, z^*+dz]$ and at the same time a *decrease* in marginal savings tax rates by $\delta\tau_s$ in the corresponding bandwidth $[s(z^*), s(z^*+dz)]$. Let construct these reforms such that they leave individuals above z^*+dz unaffected by setting the two lump-sum changes equal, meaning $d\tau_z dz = d\tau_s ds$ i.e. $d\tau_z = d\tau_s\frac{ds}{dz} = d\tau_s\,s'(z^*)$. Applying Lemma 1, the total impact of the reform on the government's tax revenue is

$$\begin{aligned}\frac{1}{\lambda}\frac{d\mathcal{L}}{dz} &= \mathcal{T}_s'\left(s^*,z^*\right)\frac{s^*}{1+\mathcal{T}_s'\left(s^*,z^*\right)}\zeta_{s|z}^c(z^*)d\tau_s\,h_z(z^*) \\ &\quad - \Big(\mathcal{T}_z'(s^*,z^*)+s_{inc}'(z^*)\mathcal{T}_s'(s^*,z^*)\Big)\frac{z^*}{1-\mathcal{T}_z'(s^*,z^*)}\zeta_z^c(z^*)d\tau_s\Big(s'(z^*)-s_{inc}'(z^*)\Big)\,h_z(z^*)\end{aligned} \tag{69}$$

while agents well-being remains unchanged. When the impact of this reform is non-zero, the type of joint reform we consider delivers a Pareto-improvement over the existing tax system. Characterizing a Pareto-efficient tax system as one that cannot be reformed in a Pareto-improving way yields the above sufficient statistics characterization.

B.6 Proof of Propositions 4, 7, and 8 (characterization of simple tax systems)

For smooth tax system, the sufficient statistic characterization for optimal tax schedules (Proposition 3) and Pareto-efficiency (Corollary 1) are obtained from local reforms of marginal earnings tax rates and marginal savings tax rates.

Because all types of simple tax systems we consider (SL, SN, and LED) feature a nonlinear earnings tax schedule, such local reforms of marginal earnings tax rates are available. As a result, the derivation of optimal earnings tax formulas for simple tax systems exactly parallels that of general smooth tax systems and the optimal marginal earnings tax rate formula (16) continues to hold. This proves Proposition 8.

Moreover, SN tax systems also feature a nonlinear savings tax schedule meaning that local reforms of marginal savings tax rates are available as well. As a result, the derivation of optimal savings tax formulas for SN tax systems exactly parallels that of general smooth tax systems, and the optimal marginal savings tax rate formula (17) continues to hold. This implies that the Pareto-efficiency formula (18) also continues to hold, proving all sufficient statistics characterizations for SN tax systems.

In contrast, local reforms of marginal savings tax rates are not available under SL and LED tax systems which assume a linear savings tax rate. For SL and LED tax systems, we thus derive optimal savings tax formulas to prove Proposition 7 and Pareto-efficiency formulas to prove Proposition 4.

B.6.1 SL tax system

Impact of a SL tax reform. When the government uses a linear savings tax such that $\mathcal{T}(s,z) = \tau_s\, s + T_z(z)$, we consider a small reform of the linear savings tax rate τ_s that consists in a small increase $d\tau_s$. Formally, this corresponds to a reform $dT_s(s) = d\tau_s\, s$ of the savings tax function $T_s(s) = \tau_s\, s$.

We characterize the impact of this reform on the government objective function. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects:*

$$\int_z \left(1 - \frac{\alpha(z)}{\lambda} U_c'\left(c(z), s(z), z; \theta(z)\right)\right) d\tau_s\, s\,(z)\; dH_z(z) \tag{70}$$

- *behavioral effects from earnings changes:*[18]

$$-\int_z T_z'(z)\left[\frac{z\zeta_z^c(z)}{1-T_z'(z)}\,d\tau_s\,s_{inc}'(z)+\frac{\eta_z(z)}{1-T_z'(z)}\,d\tau_s\,s(z)\right]\,dH_z(z) \tag{72}$$

- *behavioral effects from savings changes:*

$$-\int_z \tau_s\left[\frac{s(z)\zeta_{s|z}^c(z)}{1+\tau_s}d\tau_s+\frac{\eta_{s|z}(z)}{1+\tau_s}d\tau_s\,s(z)\right]\,dH_z(z)$$
$$-\int_z \tau_s s_{inc}'(z)\left[\frac{z\zeta_z^c(z)}{1-T_z'}\,d\tau_s\,s_{inc}'(z)+\frac{\eta_z(z)}{1-T_z'}\,d\tau_s\,s(z)\right]\,dH_z(z) \tag{73}$$

Summing over these different effects yields the total impact of the reform

$$\frac{d\mathcal{L}}{\lambda}=\int_z\left\{(1-\hat{g}(z))\,s(z)-\frac{T_z'(z)+s_{inc}'(z)\tau_s}{1-T_z'(z)}z\zeta_z^c(z)\,s_{inc}'(z)-\frac{\tau_s}{1+\tau_s}s(z)\zeta_{s|z}^c(z)\right\}d\tau_s\,dH_z(z), \tag{74}$$

with social marginal welfare weights augmented with the fiscal impact of income effects given by

$$\hat{g}(z)=\frac{\alpha(z)}{\lambda}U_c'(c(z),s(z),z;\theta(z))+\frac{T_z'(z)}{1-T_z'(z)}\eta_z(z)+\tau_s\left[\frac{\eta_{s|z}(z)}{1+\tau_s}+s_{inc}'(z)\frac{\eta_z(z)}{1-T_z'}\right].$$

Optimal savings tax formula. A direct implication of this result is a sufficient statistics characterization of the optimal linear savings tax schedule τ_s, given an existing and potentially suboptimal nonlinear tax $T_z(z)$. Indeed, at the optimum the reform should have a zero impact on the government objective meaning that the optimal savings tax schedule verifies

$$\frac{\tau_s}{1+\tau_s}\int_z s(z)\zeta_{s|z}^c(z)dH_z(z)=\int_z\left\{(1-\hat{g}(z))\;s(z)-\frac{T_z'(z)+\tau_s\,s_{inc}'(z)}{1-T_z'(z)}z\zeta_z^c(z)\,s_{inc}'(z)\right\}dH_z(z), \tag{75}$$

which is Equation (32) in Proposition 7.

Pareto-efficiency formula. To derive Pareto-efficiency, we combine savings tax and earnings tax reforms in a way that anihilates all lump-sum changes in tax liability in order to offset all utility changes.

We start with a small reform of the linear savings tax rate τ_s that consists in small increase $d\tau_s$. At the bottom of the earnings distribution ($z=\underline{z}$), the mechanical effect of the savings tax reform is an increase in tax liability by $s(\underline{z})\,d\tau_s$. We thus adjust the earnings tax liability through a downward shift by $s(\underline{z})\,d\tau_s$ at all earnings levels. This joint reform has the following impact on

[18] Applying Lemma 1, earnings and savings changes are here given by

$$\begin{cases} dz=-\frac{z\zeta_z^c(z)}{1-T_z'(z)}\,d\tau_s\,s_{inc}'(z)-\frac{\eta_z(z)}{1-T_z'(z)}\,d\tau_s\,s(z)\\ ds=-\frac{s(z)\zeta_{s|z}^c(z)}{1+\tau_s}\,d\tau_s-\frac{\eta_{s|z}(z)}{1+\tau_s}d\tau_s\;s(z)+s_{inc}'(z)dz\end{cases} \tag{71}$$

the government objective

$$\frac{d\mathcal{L}}{\lambda}=\int_{z=\underline{z}}^{\overline{z}}\left\{[1-\hat{g}(z)]\,\left[s\left(z\right)-s\left(\underline{z}\right)\right]-\frac{T_z'(z)+\tau_s\,s_{inc}'(z)}{1-T_z'}z\zeta_z^c(z)\,s_{inc}'(z)-\tau_s\frac{s(z)}{1+T_s'}\zeta_{s|z}^c(z)\right\}d\tau_s\,dH_z(z) \tag{76}$$

meaning that the lump-sum change in tax liability is nil at earnings $z=\underline{z}$ but not at earnings above $z\geq\underline{z}$.

To cancel out lump-sum changes in tax liability at all earnings levels, we are going to construct a sequence of earnings tax reforms. We discretize the range of earnings $[\underline{z},\overline{z}]$ into N bins and consider reforms in the small earnings bandwidths $dz=\frac{\Delta z}{N}$ where $\Delta z=\overline{z}-\underline{z}$. We proceed by induction to pin down a general formula for the tax reforms we consider:

- First, consider a decrease in the marginal earnings tax rate by $d\tau_z=s'\left(\underline{z}\right)d\tau_s$ over the bandwidth $[\underline{z},\underline{z}+dz]$. In this bandwidth, this reform (i) cancels out lump-sum changes in tax liability to a first-order approximation since $\left[s\left(\underline{z}+dz\right)-s\left(\underline{z}\right)\right]d\tau_s\approx s'\left(\underline{z}\right)dz d\tau_s$, and (ii) induces earnings responses through the change in marginal tax rates. Moreover, it decreases the lump-sum tax liability on all individuals with earnings $z\geq\underline{z}+dz$. The total impact of this sequence of reforms is then

$$\begin{aligned}\frac{d\mathcal{L}}{\lambda}=&\int_{z=\underline{z}+dz}^{\overline{z}}\left\{[1-\hat{g}(z)]\,\left[s\left(z\right)-s\left(\underline{z}\right)-s'\left(\underline{z}\right)dz\right]\right\}d\tau_s\,dH_z(z)\\&-\int_{z=\underline{z}}^{\overline{z}}\left\{\frac{T_z'(z)+\tau_s\,s_{inc}'(z)}{1-T_z'}z\zeta_z^c(z)\,s_{inc}'(z)-\tau_s\frac{s(z)}{1+\tau_s}\zeta_{s|z}^c(z)\right\}d\tau_s\,dH_z(z)\\&+\int_{z=\underline{z}}^{\underline{z}+dz}\frac{T_z'(z)+s_{inc}'(z)\tau_s}{1-T_z'\left(z\right)}z\zeta_z^c(z)\left(s'\left(\underline{z}\right)d\tau_s\right)\,dH_z(z)\end{aligned} \tag{77}$$

- Second, consider a decrease in the marginal earnings tax rate by $d\tau_z=s'\left(\underline{z}+dz\right)d\tau_s$ over the bandwidth $[\underline{z}+dz,\underline{z}+2dz]$, which again cancels out lump-sum changes in this bandwidth up to a first-order approximatince since $\left[s\left(\underline{z}+2dz\right)-s\left(\underline{z}\right)-s'\left(\underline{z}\right)dz\right]\approx s'\left(\underline{z}+dz\right)dz$. The total impact of this sequence of reforms is then

$$\begin{aligned}\frac{d\mathcal{L}}{\lambda}=&\int_{z=\underline{z}+2dz}^{\overline{z}}\left\{[1-\hat{g}(z)]\,\left[s\left(z\right)-s\left(\underline{z}\right)-s'\left(\underline{z}\right)dz-s'\left(\underline{z}+dz\right)dz\right]\right\}d\tau_s\,dH_z(z)\\&-\int_{z\geq\underline{z}}\left\{\frac{T_z'(z)+\tau_s\,s_{inc}'(z)}{1-T_z'}z\zeta_z^c(z)\,s_{inc}'(z)-\tau_s\frac{s(z)}{1+\tau_s}\zeta_{s|z}^c(z)\right\}d\tau_s\,dH_z(z)\\&+\int_{z=\underline{z}}^{\underline{z}+dz}\frac{T_z'(z)+s_{inc}'(z)\tau_s}{1-T_z'\left(z\right)}z\zeta_z^c(z)\left(s'\left(\underline{z}\right)d\tau_s\right)\,dH_z(z)\\&+\int_{z=\underline{z}+dz}^{\underline{z}+2dz}\frac{T_z'(z)+s_{inc}'(z)\tau_s}{1-T_z'\left(z\right)}z\zeta_z^c(z)\left(s'\left(\underline{z}+dz\right)d\tau_s\right)\,dH_z(z)\end{aligned} \tag{78}$$

- Iterating over to step k, in which we consider a decrease in the marginal earnings tax rate by $d\tau_z=s'\left(\underline{z}+(k-1)\frac{\Delta z}{N}\right)d\tau_s$ over the bandwidth $[\underline{z}+(k-1)\frac{\Delta z}{N},\underline{z}+k\frac{\Delta z}{N}]$. The total impact of this sequence of reforms is then

$$\frac{d\mathcal{L}}{\lambda} = \int_{z=\underline{z}+k\frac{\Delta z}{N}}^{\bar{z}} \left\{ [1-\hat{g}(z)] \left[s\left(z\right) - s\left(\underline{z}\right) - \frac{\Delta z}{N} \left[\sum_{p=0}^{k-1} s'\left(\underline{z} + p\frac{\Delta z}{N} \right) \right] \right] \right\} d\tau_s\, dH_z(z)$$
$$- \int_{z \geq \underline{z}} \left\{ \frac{T_z'(z) + \tau_s\, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s} \zeta_{s|z}^c(z) \right\} d\tau_s\, dH_z(z)$$
$$+ \sum_{p=0}^{k-1} \int_{z=\underline{z}+p\frac{\Delta z}{N}}^{\underline{z}+(p+1)\frac{\Delta z}{N}} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) s'\left(\underline{z} + p\frac{\Delta z}{N} \right) d\tau_s\, dH_z(z) \quad (79)$$

- Pushing the iteration forward until $k = N$, the first integral disappears (integration over an empty set) such that the total impact of this sequence of reforms is given by

$$\frac{d\mathcal{L}}{\lambda} = - \int_{z \geq \underline{z}} \left\{ \frac{T_z'(z) + \tau_s\, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s} \zeta_{s|z}^c(z) \right\} d\tau_s\, dH_z(z)$$
$$+ \sum_{p=0}^{N-1} \int_{z=\underline{z}+p\frac{\Delta z}{N}}^{\underline{z}+(p+1)\frac{\Delta z}{N}} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) s'\left(\underline{z} + p\frac{\Delta z}{N} \right) d\tau_s\, dH_z(z) \quad (80)$$

Let's now compute the last term at the limit $N \to \infty$. We have

$$\sum_{p=0}^{N-1} \int_{z=\underline{z}+p\frac{\Delta z}{N}}^{\underline{z}+(p+1)\frac{\Delta z}{N}} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z) s'\left(\underline{z} + p\frac{\Delta z}{N} \right) d\tau_s\, dH_z(z) \quad (81)$$

$$\approx \sum_{p=0}^{N-1} \frac{T_z'(\underline{z}+p\frac{\Delta z}{N}) + s_{inc}'(\underline{z}+p\frac{\Delta z}{N})\tau_s}{1-T_z'\left(\underline{z}+p\frac{\Delta z}{N}\right)} \left(\underline{z}+p\frac{\Delta z}{N}\right) \zeta_z^c \left(\underline{z}+p\frac{\Delta z}{N}\right) s'\left(\underline{z}+p\frac{\Delta z}{N}\right) d\tau_s h_z \left(\underline{z}+p\frac{\Delta z}{N}\right) \frac{\Delta z}{N} \quad (82)$$

$$\underset{N\to\infty}{\longrightarrow} \int_{z=\underline{z}}^{\bar{z}} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z)\, s'(z)\, d\tau_s\, h_z(z)\, dz \quad (83)$$

where the last line follows from the Riemann definition of the integral in terms of Riemann sums. Hence, the total impact of this sequence of reforms is at the limit given by

$$\frac{d\mathcal{L}}{\lambda} = - \int_{z \geq \underline{z}} \left\{ \frac{T_z'(z) + \tau_s\, s_{inc}'(z)}{1-T_z'} z\zeta_z^c(z)\, s_{inc}'(z) - \tau_s \frac{s(z)}{1+\tau_s} \zeta_{s|z}^c(z) \right\} d\tau_s\, h_z(z) dz$$
$$+ \int_{z=\underline{z}}^{\bar{z}} \frac{T_z'(z) + s_{inc}'(z)\tau_s}{1-T_z'(z)} z\zeta_z^c(z)\, s'(z)\, d\tau_s\, h_z(z)\, dz \quad (84)$$

By construction, the sequence of reforms we have constructed does not affect agents' utility, and only affects tax revenue through the expression above. When the impact of this reform is non-zero, the type of sequence of reforms we consider delivers a Pareto-improvement over the existing tax system. Characterizing a Pareto-efficient tax system as one that cannot be reformed in a

Pareto-improving way yields the following sufficient statistics characterization:

$$\frac{\tau_s}{1+\tau_s}\int_{z=\underline{z}}^{\bar{z}} s(z)\zeta_{s|z}^c(z)h_z(z)dz = \int_{z=\underline{z}}^{\bar{z}} \left[s'(z) - s'_{inc}(z)\right]\frac{T'_z(z) + s'_{inc}(z)\tau_s}{1-T'_z(z)} z\zeta_z^c(z)\, h_z(z)\, dz, \tag{85}$$

which is Equation (19) in Proposition 4.

B.6.2 Linear earnings-dependent savings tax

Earnings responses to LED savings tax reform We begin by translating Lemma 1 for LED tax systems, where we consider particular savings tax reforms.

For a LED tax system, we consider savings tax reform $dT_s(s|z)$ that consists in a small increase $d\tau_s$ of the savings linear tax rate phased-in over the earnings bandwidth $[z^*, z^* + dz]$. Formally, this corresponds to the following reform $dT_s(s|z)$ of the savings tax function $T_s(s|z) = \tau_s(z)\, s$:

$$dT_s(s|z) = \begin{cases} 0 & if\ z \leq z^* \\ d\tau_s(z-z^*)\, s & if\ z \in [z^*, z^*+dz] \\ d\tau_s dz\, s & if\ z \geq z^* + dz \end{cases} \tag{86}$$

Now, let

$$V(T_s(.|z), y, z; \theta) = \max_s\ U(y - s - T_s(s|z), s, z; \theta) \tag{87}$$

be agent θ indirect utility function at earnings z and (pre savings tax) disposable income y.

The idea is to construct for each type θ a perturbation of the earnings tax $dT_z^\theta(z)$ that induces the same earnings response as the initial perturbation $dT_s(s|z)$. Suppose we define this perturbation for each type θ such that at all earnings z,

$$V(T_s(.) + dT_s(.), z - T_z(z), z; \theta) = V(T_s(.), z - T_z(z) - dT_z^\theta(z), z; \theta) \tag{88}$$

then, by construction, dT_z^θ induces the same earnings response as the initial perturbation dT_s. Moreover, dT_z^θ induces the same utility change as the initial perturbation dT_s.

Applying the envelope theorems to agents in the earnings bandwidth $[z^*, z^* + dz]$ yields

$$U'_c \cdot d\tau_s(z-z^*)\, s(z;\theta) = U'_c \cdot dT_z^\theta(z)$$
$$\iff dT_z^\theta(z) = d\tau_s(z-z^*)\, s(z;\theta)$$

As can thus be seen differentiating both sides by earnings z, a small increase by $d\tau_s$ in the marginal savings tax rate phased-in over the earnings bandwidth $[z^*, z^*+dz]$ leads for agents in this bandwidth to the same earnings responses as a small increase by $s(z;\theta)\, d\tau_s + s'_{inc}(z)(z-z^*)\, d\tau_s$ in the marginal earnings tax rate. Noting that $s'_{inc}(z)(z-z^*)\, d\tau_s$ is second-order, and vanishes when evaluated at $z = z^*$, the first-order equivalent earnings tax reform is therefore a $s(z^*)\, d\tau_s$ increase in the marginal earnings tax rate.

Applying the envelope theorems to agents above the earnings bandwidth $[z^*, z^* + dz]$ yields

$$dT_z^\theta(z) = d\tau_s dz\, s\,(z;\theta)$$

such that the equivalent earnings tax reform is a $d\tau_s dz\, s'_{inc}\,(z)$ increase in the marginal earnings tax rate.

Impact of a LED savings tax reform. When the government uses a linear earnings-dependent savings tax such that $\mathcal{T}\,(z,s) = T_z\,(z) + \tau_s\,(z)\ s$, we consider a small reform of the linear earnings-dependent savings tax $\tau_s(z)$ that consists in a small increase $\delta\tau_s$ of the savings linear tax rate phased-in over the earnings bandwidth $[z^*, z^* + dz]$. This means that agents in this bandwidth face a linear tax rate equal to $\tau_s\,(z) + d\tau_s\,(z - z^*)$ and that all agents above face a linear tax rate equal to $\tau_s\,(z) + d\tau_s dz$. Formally, this corresponds to the following reform $dT_s(s|z)$ of the savings tax function $T_s\,(s|z) = \tau_s\,(z)\ s$:

$$dT_s(s|z) = \begin{cases} 0 & if\ z \leq z^* \\ d\tau_s\,(z - z^*)\,s & if\ z \in [z^*, z^* + dz] \\ d\tau_s dz\, s & if\ z \geq z^* + dz \end{cases} \tag{89}$$

We characterize the impact of this reform on the government objective function. Normalizing all effects by $1/\lambda$, the reform induces

- *mechanical effects:*

$$\int_{z \geq z^*} \left(1 - \frac{\alpha(z)}{\lambda} U'_c\,(c(z), s(z), z; \theta(z))\right) d\tau_s dz\, s\,(z)\ dH_z(z) \tag{90}$$

- *behavioral effects from earnings changes*:[19]

$$\begin{aligned} &- \mathcal{T}'_z\,(s\,(z^*)\,, z^*) \left[\frac{z^* \zeta_z^c(z^*)}{1 - \mathcal{T}'_z\,(s\,(z^*)\,, z^*)}\, d\tau_s s(z^*)\right] h_z(z^*) dz \\ &- \int_{z^*}^{\bar{z}} \mathcal{T}'_z\,(s\,(z)\,, z) \left[\frac{z \zeta_z^c(z)}{1 - \mathcal{T}'_z\,(s\,(z)\,, z)}\, s'_{inc}(z) + \frac{\eta_z(z)}{1 - \mathcal{T}'_z\,(s\,(z)\,, z)}\, s(z)\right] d\tau_s dz\, dH_z(z) \end{aligned} \tag{91}$$

[19] Applying Lemma 1, earnings and savings changes at earnings z^* and at earnings above z^* are respectively given by

$$\begin{cases} dz = -\frac{z^* \zeta_z^c(z^*)}{1 - \mathcal{T}'_z}\, d\tau_s\, s(z^*) \\ ds = s'_{inc}(z^*) dz \end{cases} \quad \text{and} \quad \begin{cases} dz = -\frac{z \zeta_z^c(z)}{1 - \mathcal{T}'_z}\, d\tau_s dz\, s'_{inc}(z) - \frac{\eta_z(z)}{1 - \mathcal{T}'_z}\, d\tau_s dz\, s\,(z) \\ ds = -\frac{s(z) \zeta^c_{s|z}(z)}{1 + \tau_s(z)}\, d\tau_s dz - \frac{\eta_s(z)}{1 + \tau_s(z)} d\tau_s dz\ s\,(z) + s'_{inc}(z) dz \end{cases}$$

- *behavioral effects from savings changes*:

$$
\begin{aligned}
&-\tau_s(z^*)s'_{inc}(z^*)\left[\frac{z^*\zeta_z^c(z^*)}{1-\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)}\,d\tau_s\,s(z^*)\right]h_z(z^*)dz\\
&-\int_{z^*}^{\bar{z}}\tau_s(z)\left[\frac{\zeta_{s|z}^c(z)+\eta_{s|z}(z)}{1+\tau_s(z)}\,s(z)+s'_{inc}(z)\left(\frac{z\zeta_z^c(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}s'_{inc}(z)+\frac{\eta_z(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}\,s(z)\right)\right]d\tau_s dz\,dH_z(z)
\end{aligned}
\tag{92}
$$

Summing over these different effects yields the total impact of the reform

$$
\begin{aligned}
\frac{1}{\lambda}\frac{d\mathcal{L}}{dz}=&\int_{z^*}^{\bar{z}}\left\{\left(1-\hat{g}(z)\right)s\left(z\right)-\frac{\mathcal{T}_z'\left(s\left(z\right),z\right)+s'_{inc}(z)\tau_s(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}z\zeta_z^c(z)s'_{inc}(z)-\frac{\tau_s(z)}{1+\tau_s(z)}s(z)\zeta_{s|z}^c(z)\right\}d\tau_s\,dH_z(z)\\
&-\frac{\mathcal{T}_z'\left(z^*,s\left(z^*\right)\right)+s'_{inc}(z^*)\tau_s(z^*)}{1-\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)}z^*\zeta_z^c(z^*)\,s(z^*)d\tau_s\,h_z(z^*)
\end{aligned}
\tag{93}
$$

with

$$
\hat{g}(z)=\frac{\alpha(z)}{\lambda}U_c'\left(c(z),s(z),z;\theta(z)\right)-\frac{\mathcal{T}_z'\left(s\left(z\right),z\right)+s'_{inc}(z)\tau_s(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}\eta_z(z)-\frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(z).
$$

Optimal savings tax formula. A direct implication of this result is a sufficient statistics characterization of the optimal linear earnings-dependent savings tax $\tau_s(z)$, given an existing and potentially suboptimal nonlinear earnings tax $T_z\left(z\right)$. Indeed, at the optimum the reform should have a zero impact on the government objective, meaning that the optimal linear earnings-dependent savings tax schedule verifies at each earnings z^*

$$
\begin{aligned}
&\frac{\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)+s'_{inc}(z^*)\tau_s(z^*)}{1-\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)}z^*\zeta_z^c(z^*)\,s(z^*)\,h_z(z^*)\\
&=\int_{z^*}^{\bar{z}}\left\{\left(1-\hat{g}(z)\right)s\left(z\right)-\frac{\mathcal{T}_z'\left(s\left(z\right),z\right)+s'_{inc}(z)\tau_s(z)}{1-\mathcal{T}_z'\left(s\left(z\right),z\right)}z\zeta_z^c(z)s'_{inc}(z)-\frac{\tau_s(z)}{1+\tau_s(z)}s(z)\zeta_{s|z}^c(z)\right\}dH_z(z),
\end{aligned}
\tag{94}
$$

which is Equation (31) in Proposition 7.

Pareto-efficiency formula. To derive Pareto-efficiency, we combine savings tax and earnings tax reforms in a way that anihilates all lump-sum changes in tax liability in order to offset all utility changes. To do so, consider the following sequence of reforms.

First, consider an *increase* $d\tau_s$ of the income-contingent savings tax $\tau_s(z)$ by phased-in over the earnings bandwidth $[z^*, z^*+dz]$, together with a *decrease* $d\tau_z$ of the marginal tax rate on earnings in the same bandwidth. Imposing $d\tau_z=s(z^*)d\tau_s$, this joint reform cancels out substitution effects at z^* and has the following impact on the government objective:

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{dz} = \int_{z^*}^{\bar{z}} \left\{ (1-\hat{g}(z))\, s(z) - \frac{\mathcal{T}_z'(s(z),z) + s_{inc}'(z)\tau_s(z)}{1-\mathcal{T}_z'(s(z),z)} z\zeta_z^c(z) s_{inc}'(z) - \frac{\tau_s(z)}{1+\tau_s(z)} s(z)\zeta_{s|z}^c(z) \right\} d\tau_s\, dH_z(z) - \int_{z^*}^{\bar{z}} (1-\hat{g}(z))\; s(z^*) d\tau_s\, dH_z(z) \tag{95}$$

Second, consider a *decrease* by $d\tau_s$ of the income-contingent savings tax $\tau_s(z)$ phased-in over the earnings bandwidth $[z^*+dz, z^*+2dz]$, together with an *increase* $d\tau_z$ of the marginal tax rate on earnings in the same bandwidth. Imposing $d\tau_z = s(z^*+\delta z)d\tau_s$, this joint reform cancels out substitution effects at z^* and has the following impact on the government objective:

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{dz} = -\int_{z^*+dz}^{\bar{z}} \left\{ (1-\hat{g}(z))\, s(z) - \frac{\mathcal{T}_z'(s(z),z) + s_{inc}'(z)\tau_s(z)}{1-\mathcal{T}_z'(s(z),z)} z\zeta_z^c(z) s_{inc}'(z) - \frac{\tau_s(z)}{1+\tau_s(z)} s(z)\zeta_{s|z}^c(z) \right\} d\tau_s\, dH_z(z) + \int_{z^*+dz}^{\bar{z}} (1-\hat{g}(z))\; s(z^*+dz) d\tau_s\, dH_z(z) \tag{96}$$

The total impact of these two joint reforms on the objective function of the government is thus

$$\frac{d\mathcal{L}}{\lambda} = \int_{z^*}^{z^*+dz} \left\{ (1-\hat{g}(z))\, s(z) - \frac{\mathcal{T}_z'(s(z),z) + s_{inc}'(z)\tau_s(z)}{1-\mathcal{T}_z'(s(z),z)} z\zeta_z^c(z) s_{inc}'(z) - \frac{\tau_s(z)}{1+\tau_s(z)} s(z)\zeta_{s|z}^c(z) \right\} d\tau_s dz\, dH_z(z) + [s(z^*+dz) - s(z^*)] \int_{z^*}^{\bar{z}} (1-\hat{g}(z))\, d\tau_s dz dH_z(z) - s(z^*) \int_{z^*}^{z^*+dz} (1-\hat{g}(z))\, d\tau_s dz dH_z(z) \tag{97}$$

where at the limit $dz \to 0$, the term $(1-\hat{g}(z))\, s(z)$ in the integral on the first line excatly cancels out with the last integral term on the second line.

Third, we consider a *decrease* by $d\tau_z$ of the marginal tax rate on earnings in the bandwidth $[z^*, z^*+dz]$. Imposing $d\tau_z = [s(z^*+dz) - s(z^*)]\, d\tau_s$, this reform cancels out the other integral term on the second line and generates earnings responses at the margin. As a result, the total impact of this sequence of reforms is at the limit $dz \to 0$ given by

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{dz} \to \left\{ -s_{inc}'(z^*) \frac{\mathcal{T}_z'(s(z^*),z^*) + s_{inc}'(z^*)\tau_s(z^*)}{1-\mathcal{T}_z'(s(z^*),z^*)} z^*\zeta_z^c(z^*) - \frac{\tau_s(z^*)}{1+\tau_s(z^*)} s(z^*)\zeta_{s|z}^c(z^*) \right\} d\tau_s\, h_z(z^*)dz + s'(z^*) \frac{\mathcal{T}_z'(s(z^*),z^*) + s_{inc}'(z^*)\tau_s(z^*)}{1-\mathcal{T}_z'(s(z^*),z^*)} z^*\zeta_z^c(z^*) d\tau_s\, h_z(z^*)dz \tag{98}$$

where the $s'(z^*)$ in the last line comes from the fact that $[s(z^*+dz) - s(z^*)]\, d\tau_s \to s'(z^*)\, dz\, d\tau_s$. This expression is the impact of this sequence of reforms on the government's tax revenue, while agents well-being remains unchanged. When this impact is non-zero, the type of reforms we consider delivers a Pareto-improvement over the existing tax system. Characterizing a Pareto-efficient tax system as one that cannot be reformed in a Pareto-improving way yields the following sufficient statistics characterization:

$$\frac{\tau_s(z^*)}{1+\tau_s(z^*)}s(z^*)\zeta^c_{s|z}(z^*) = \left[s'(z^*) - s'_{inc}(z^*)\right]\frac{\mathcal{T}'_z\left(s\left(z^*\right),z^*\right) + s'_{inc}(z^*)\tau_s(z^*)}{1-\mathcal{T}'_z\left(s\left(z^*\right),z^*\right)}z^*\zeta^c_z(z^*), \tag{99}$$

which is Equation (21) in Proposition 4.

B.7 Proof of Proposition 5 (simple tax systems and multidimensional heterogeneity)

We characterize in Proposition 5 optimal savings tax formulas for each type of simple tax system in the presence of multidimensional heterogeneity. These formulas take the actual earnings tax schedule as given, be they optimally set or not, and extend the results derived in the unidimensional case. Crucially, we are able to provide similar characterizations because Lemma 1 still holds in the presence of multidimensional heterogeneity.

B.7.1 Separable linear (SL) tax system

Consider a reform that consists in a $d\tau_s$ increase in the linear savings tax rate τ_s. For all agents, this triggers an increase in tax liability by $s\,d\tau_s$ and an increase in the marginal savings tax rate by $d\tau_s$ – which by Lemma 1 produces earnings responses equivalent to an increase in the marginal earnings tax rate by $s'_{inc}d\tau_s$.

This reform has the following effects on the government objective:

- mechanical effects

$$\int_z\int_s\Big[(1-g\left(s,z\right))\,s\,d\tau_s\Big]h\left(s,z\right)dsdz \tag{100}$$

$$=\int_z \mathbb{E}\left[(1-g\left(s,z\right))\,s|z\right]\,d\tau_s h_z\left(z\right)dz \tag{101}$$

- earnings responses[20]

$$\int_z T'_z\left(z\right)\left\{\int_s\left(-\frac{z}{1-T'_z\left(z\right)}\zeta^c_z(s,z)\,s'_{inc}\left(s,z\right)d\tau_s - \frac{\eta_z(s,z)}{1-T'_z\left(z\right)}s\,d\tau_s\right)h\left(s,z\right)ds\right\}dz \tag{102}$$

$$=-\int_z\frac{T'_z\left(z\right)}{1-T'_z\left(z\right)}\left\{\mathbb{E}\left[z\zeta^c_z(s,z)s'_{inc}\left(s,z\right)+\eta_z(s,z)s|z\right]\right\}d\tau_s h_z\left(z\right)dz \tag{103}$$

[20]We have

$$dz = \frac{\partial z}{\partial T'_z}\delta T'_z + \frac{\partial z}{\partial T_z}\delta T_z = -\frac{z}{1-T'_z}\zeta^c_z(z)\,\delta T'_z - \frac{\eta_z(z)}{1-T'_z}\delta T_z$$

- savings responses[21]

$$\tau_s \int_z \int_s \Bigg\{ -\frac{s}{1+\tau_s}\zeta^c_{s|z}(s,z)\, d\tau_s - \frac{\eta_{s|z}(s,z)}{1+\tau_s}\, s\, d\tau_s \tag{104}$$
$$+ s'_{inc}(s,z)\left(-\frac{z}{1-T'_z(z)}\zeta^c_z(s,z)\, s'_{inc}(s,z)\, d\tau_s - \frac{\eta_z(s,z)}{1-T'_z(z)} s\, d\tau_s \right)\Bigg\} h(s,z)\, ds dz$$
$$= -\tau_s \int_z \Bigg\{ \frac{1}{1+\tau_s}\mathbb{E}\left[s\zeta^c_{s|z}(s,z) + \eta_{s|z}(s,z)\, s | z \right] \tag{105}$$
$$+ \frac{1}{1-T'_z(z)}\left(\mathbb{E}\left[z\zeta^c_z(s,z)\ \left(s'_{inc}(s,z)\right)^2 + \eta_z(s,z)\, s\, s'_{inc}(s,z)\, |z \right] \right)\Bigg\} d\tau_s h_z(z)\, dz$$

Such that the total impact of the reform on the government objective is

$$\frac{d\mathcal{L}}{d\tau_s} = \int_z \mathbb{E}\left[(1-g(s,z))\, s|z \right]\ h_z(z)\, dz \tag{106}$$
$$-\int_z \frac{T'_z(z)}{1-T'_z(z)}\Bigg\{ \mathbb{E}\left[z\zeta^c_z(s,z) s'_{inc}(s,z) + \eta_z(s,z)\, s|z \right] \Bigg\} h_z(z)\, dz \tag{107}$$
$$-\tau_s \int_z \Bigg\{ \frac{1}{1+\tau_s}\mathbb{E}\left[s\zeta^c_{s|z}(s,z) + \eta_{s|z}(s,z)\, s|z \right]$$
$$+ \frac{1}{1-T'_z(z)}\left(\mathbb{E}\left[z\zeta^c_z(s,z)\ \left(s'_{inc}(s,z)\right)^2 + \eta_z(s,z)\, s\, s'_{inc}(s,z)\, |z \right] \right)\Bigg\} h_z(z)\, dz \tag{108}$$

Redefining augmented social marginal welfare weights as

$$\hat{g}(s,z) = g(s,z) + \frac{T'_z(z)}{1-T'_z(z)}\eta_z(s,z) + \frac{\tau_s}{1+\tau_s}\eta_{s|z}(s,z) + \frac{\tau_s}{1-T'_z(z)}\eta_z(s,z) s'_{inc}(s,z) \tag{109}$$

we finally get

$$\frac{d\mathcal{L}}{d\tau_s} = \int_z \mathbb{E}\left[(1-\hat{g}(s,z))\, s|z \right]\ h_z(z)\, dz - \int_z \frac{T'_z(z)}{1-T'_z(z)}\Bigg\{ \mathbb{E}\left[z\zeta^c_z(s,z) s'_{inc}(s,z)\, |z \right] \Bigg\} h_z(z)\, dz \tag{110}$$
$$-\tau_s \int_z \Bigg\{ \frac{1}{1+\tau_s}\mathbb{E}\left[s\zeta^c_{s|z}(s,z)|z \right] + \frac{1}{1-T'_z(z)}\left(\mathbb{E}\left[z\zeta^c_z(s,z)\ \left(s'_{inc}(s,z)\right)^2 |z \right] \right)\Bigg\} h_z(z)\, dz$$

The optimal savings linear tax rate τ_s thus satisfies

$$\frac{\tau_s}{1+\tau_s}\int_z \Bigg\{ \mathbb{E}\left[s\zeta^c_{s|z}(s,z)|z \right] \Bigg\} dH_z(z) \tag{111}$$
$$= \int_z \Bigg\{ \mathbb{E}\left[(1-\hat{g}(s,z))\, s|z \right] - \mathbb{E}\left[\frac{T'_z(z) + s'_{inc}(s,z)\,\tau_s}{1-T'_z(z)} z\zeta^c_z(s,z) s'_{inc}(s,z)\, |z \right] \Bigg\} dH_z(z)$$

[21] We have
$$ds = \frac{\partial s}{\partial T'_s}\delta T'_s + \frac{\partial s}{\partial T_s}\delta T_s + \frac{ds(z;\theta)}{dz}dz = -\frac{s}{1+T'_s}\zeta^c_{s|z}\,\delta T'_s - \frac{\eta_{s|z}(z)}{1+T'_s}\delta T_s + s'_{inc}(z)dz$$

B.7.2 Separable nonlinear (SN) tax system

Consider a reform that consists in a $\delta\tau_s$ increase in the marginal savings tax rate across the savings bandwidth $[s^*, s^*+\delta s]$. For all agents with savings above s^*, this triggers an increase in tax liability by $\delta s\,\delta\tau_s$. For agents at s^*, this triggers an increase in the marginal savings tax rate by $\delta\tau_s$ – which by Lemma 1 produces earnings responses equivalent to an increase in the marginal earnings tax rate by $s'_{inc}\delta\tau_s$.

This reform has the following effects on the government objective:

- mechanical effects

$$\begin{aligned}&\int_{s\geq s^*}\int_z\Big\{(1-g(s,z))\,\delta s\delta\tau_s\Big\}h(s,z)\,dsdz \\ &=\int_z\Big\{\mathbb{E}\left[1-g(s,z)\,|z,s\geq s^*\right]\Big\}\delta s\delta\tau_s h_z(z)\,dz\end{aligned} \tag{112}$$

- earnings responses[22]

$$\begin{aligned}&-\int_z T'_z(z)\left\{\frac{z}{1-T'_z}\zeta^c_z(s^*,z)\,s'_{inc}(s^*,z)\,\delta\tau_s\right\}\delta s\,h(s^*,z)\,dz \\ &-\int_{s\geq s^*}\int_z T'_z(z)\left\{\frac{\eta_z(s,z)}{1-T'_z(z)}\,\delta\tau_s\delta s\right\}h(s,z)\,dsdz \\ &=-\int_z\frac{T'_z(z)}{1-T'_z}\Big\{z\zeta^c_z(s^*,z)\,s'_{inc}(s^*,z)+\mathbb{E}\left[\eta_z(s,z)s|z,s\geq s^*\right]\Big\}\delta\tau_s\delta s\,h_z(z)\,dz\end{aligned} \tag{113}$$

- savings responses[23]

$$-T'_s(s^*)\int_z\left\{\frac{s^*}{1+T'_s(s^*)}\zeta^c_{s|z}(s^*,z)\,\delta\tau_s+s'_{inc}(s^*,z)\,\frac{z}{1-T'_z(z)}\zeta^c_z(s^*,z)\,s'_{inc}(s^*,z)\,\delta\tau_s\right\}\delta s\,h(s^*,z)\,dz \tag{114}$$

$$-\int_{s\geq s^*}\int_z\left\{T'_s(s)\left(\frac{\eta_{s|z}(s,z)}{1+T'_s(s)}\delta s\delta\tau_s+s'_{inc}(s,z)\,\frac{\eta_z(s,z)}{1-T'_z(z)}\delta s\delta\tau_s\right)\right\}h(s,z)\,dsdz$$

$$\begin{aligned}&=-\int_z\left\{\frac{T'_s(s^*)}{1+T'_s(s^*)}s^*\zeta^c_{s|z}(s^*,z)+\frac{T'_s(s^*)}{1-T'_z(z)}s'_{inc}(s^*,z)^2\,z\zeta^c_z(s^*,z)\right\}\delta\tau_s\delta s\,h_z(z)\,dz \\ &-\int_z\left\{\mathbb{E}\left[\frac{T'_s(s)}{1+T'_s(s)}\eta_{s|z}(s,z)\middle|z,s\geq s^*\right]+\mathbb{E}\left[s'_{inc}(s,z)\,\frac{T'_s(s)}{1-T'_z(z)}\eta_z(s,z)\middle|z,s\geq s^*\right]\right\}\delta s\delta\tau_s h_z(z)\,dz\end{aligned} \tag{115}$$

[22] We have

$$dz=\frac{\partial z}{\partial T'_z}\delta T'_z+\frac{\partial z}{\partial T_z}\delta T_z=-\frac{z}{1-T'_z}\zeta^c_z(z)\,\delta T'_z-\frac{\eta_z(z)}{1-T'_z}\delta T_z$$

[23] We have

$$ds=\frac{\partial s}{\partial T'_s}\delta T'_s+\frac{\partial s}{\partial T_s}\delta T_s+\frac{ds(z;\theta)}{dz}dz=-\frac{s}{1+T'_s}\zeta^c_{s|z}\,\delta T'_s-\frac{\eta_{s|z}(z)}{1+T'_s}\delta T_s+s'_{inc}(z)dz$$

Such that the total impact of the reform on the government objective is

$$\begin{aligned}
\frac{d\mathcal{L}}{\delta s\delta\tau_s} = & \int_z \Big\{ \mathbb{E}\left[1-g\left(s,z\right)|z,s\geq s^*\right] \Big\} dH_z\left(z\right) \qquad (116)\\
& -\int_z \frac{T_z'\left(z\right)}{1-T_z'} \Big\{ z\zeta_z^c\left(s^*,z\right)s_{inc}'\left(s^*,z\right)+\mathbb{E}\left[\eta_z(s,z)|z,s\geq s^*\right] \Big\} dH_z\left(z\right)\\
& -\int_z \left\{ \frac{T_s'\left(s^*\right)}{1+T_s'\left(s^*\right)} s^*\zeta_{s|z}^c(s^*,z) + \frac{T_s'\left(s^*\right)}{1-T_z'\left(z\right)} s_{inc}'\left(s^*,z\right)^2 z\zeta_z^c(s^*,z) \right\} dH_z\left(z\right)\\
& -\int_z \left\{ \mathbb{E}\left[\frac{T_s'\left(s\right)}{1+T_s'\left(s\right)}\eta_{s|z}(s,z)\middle| z,s\geq s^*\right] + \mathbb{E}\left[s_{inc}'\left(s,z\right)\frac{T_s'\left(s\right)}{1-T_z'\left(z\right)}\eta_z(s,z)\middle| z,s\geq s^*\right]\right\} dH_z\left(z\right)
\end{aligned}$$

Redefining augmented social marginal welfare weights as

$$\hat{g}(s,z) = g(s,z) + \frac{T_z'\left(z\right)}{1-T_z'\left(z\right)}\eta_z(s,z) + \frac{T_s'\left(s\right)}{1+T_s'\left(s\right)}\eta_{s|z}(s,z) + s_{inc}'(s,z)\frac{T_s'\left(s\right)}{1-T_z'\left(z\right)}\eta_z(s,z) \qquad (117)$$

we finally get

$$\begin{aligned}
\frac{d\mathcal{L}}{\delta s\delta\tau_s} = & \int_z \Big\{ \mathbb{E}\left[1-g\left(s,z\right)|z,s\geq s^*\right] \Big\} dH_z\left(z\right) - \int_z \left\{ \frac{T_z'\left(z\right)}{1-T_z'} z\zeta_z^c\left(s^*,z\right)s_{inc}'\left(s^*,z\right) \right\} dH_z\left(z\right) \qquad (118)\\
& -\int_z \left\{ \frac{T_s'\left(s^*\right)}{1+T_s'\left(s^*\right)} s^*\zeta_{s|z}^c(s^*,z) + \frac{T_s'\left(s^*\right)}{1-T_z'\left(z\right)} s_{inc}'\left(s^*,z\right)^2 z\zeta_z^c(s^*,z) \right\} dH_z\left(z\right)
\end{aligned}$$

The optimal marginal savings tax rate $T_s'(.)$ thus satisfies at each savings s^*,

$$\begin{aligned}
& \frac{T_s'\left(s^*\right)}{1+T_s'\left(s^*\right)} \int_z \left\{ s^*\zeta_{s|z}^c(s^*,z) \right\} dH_z\left(z\right) \qquad (119)\\
& = \int_z \Big\{ \mathbb{E}\left[1-g\left(s,z\right)|z,s\geq s^*\right] \Big\} dH_z\left(z\right) - \int_z \left\{ \frac{T_z'\left(z\right)+s_{inc}'\left(s^*,z\right)T_s'\left(s^*\right)}{1-T_z'\left(z\right)} z\zeta_z^c\left(s^*,z\right)s_{inc}'\left(s^*,z\right) \right\} dH_z\left(z\right)
\end{aligned}$$

B.7.3 Linear earnings-dependent (LED) tax system

Consider a reform that consists in a $\delta\tau_s\delta z$ increase in the linear savings tax rate phased-in across the earnings bandwidth $[z^*, z^*+\delta z]$.

For all agents with earnings above z^*, this triggers an increase in the linear savings tax rate by $\delta\tau_s\delta z$ meaning that the marginal savings tax rate increases by the same magnitude – which triggers by Lemma 1 earnings responses equivalent to those induced by a $s_{inc}'\delta\tau_s\delta z$ increase in the marginal earnings tax rate – and that agents' savings tax burden increases by $s\,\delta\tau_s\delta z$.

For agents at z^*, the only direct effect of the reform is to induce earnings responses which by Lemma 1 are equivalent to an increase in the marginal earnings tax rate given by $s\,\delta\tau_s$ (as in the unidimensional case).

This reform has the following effects on the government objective:

- mechanical effects[24]

$$\int_{z\geq z^*}\int_s\Big\{(1-g(s,z))\,\delta z\delta\tau_s s\Big\}h(s,z)\,dsdz \tag{120}$$
$$=\int_{z\geq z^*}\Big\{E_s\Big[(1-g(s,z))\,s\Big|z\Big]\delta z\delta\tau_s\Big\}h_z(z)\,dz$$

- earnings responses[25]

$$-\int_s\left(T_z'(z^*)+\tau_s'(z^*)\,s\right)\left\{\frac{z^*}{1-T_z'(z^*)-\tau_s'(z^*)\,s}\zeta_z^c(s,z^*)\,s\,\delta\tau_s\right\}\delta z\,h(s,z^*)\,ds \tag{121}$$
$$-\int_{z\geq z^*}\int_s\left(T_z'(z)+\tau_s'(z)\,s\right)\left\{\frac{z}{1-T_z'(z)-\tau_s'(z)\,s}\zeta_z^c(s,z)s_{inc}'(s,z)\,\delta z\delta\tau_s+\frac{\eta_z(s,z)}{1-T_z'(z)-\tau_s'(z)\,s}\,s\,\delta z\delta\tau_s\right\}h(s,$$
$$=-\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s\Big|z=z^*\right]\delta z\delta\tau_s\,h_z(z^*) \tag{122}$$
$$-\int_{z\geq z^*}\left\{\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s}{1-T_z'(z)-\tau_s'(z)\,s}\Big(z\zeta_z^c(s,z)s_{inc}'(s,z)+s\,\eta_z(s,z)\Big)\Big|z\right]\right\}\delta z\delta\tau_s h_z(z)\,dz$$

- savings responses[26]

$$-\tau_s(z^*)\int_s s_{inc}'(s,z^*)\left\{\frac{z^*}{1-T_z'(z^*)-\tau_s'(z^*)\,s}\zeta_z^c(s,z^*)\,s\,\delta\tau_s\right\}\delta z\,h(s,z^*)\,ds \tag{123}$$
$$-\int_{z\geq z^*}\int_s\tau_s(z)\left\{\frac{s}{1+\tau_s(z)}\zeta_{s|z}^c(s,z)\,\delta z\delta\tau_s+s_{inc}'(s,z)\,\frac{z}{1-T_z'(z)-\tau_s'(z)\,s}\zeta_z^c(s,z)\,s_{inc}'(s,z)\,\delta z\delta\tau_s\right\}h(s,z)\,d$$
$$-\int_{z\geq z^*}\int_s\left\{\tau_s(z)\left(\frac{\eta_{s|z}(s,z)}{1+\tau_s(z)}\,s\,\delta z\delta\tau_s+s_{inc}'(s,z)\,\frac{\eta_z(s,z)}{1-T_z'(z)-\tau_s'(z)\,s}\,s\,\delta z\delta\tau_s\right)\right\}h(s,z)\,dsdz$$
$$=-\tau_s(z^*)\,\mathbb{E}\left[s_{inc}'(s,z)\,\frac{z\zeta_z^c(s,z)s}{1-T_z'(z)-\tau_s'(z)\,s}\Big|z=z^*\right]\delta z\delta\tau_s\,h_z(z^*) \tag{124}$$
$$-\int_{z\geq z^*}\left\{\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c(s,z)\Big|z\right]+\mathbb{E}\left[\frac{\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s_{inc}'(s,z)^2\right]\right\}\delta z\delta\tau_s\,h_z(z)\,dz$$
$$-\int_{z\geq z^*}\left\{\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\eta_{s|z}(s,z)\Big|z\right]+\mathbb{E}\left[\frac{\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}s\eta_z(s,z)s_{inc}'(s,z)\Big|z\right]\right\}\delta z\delta\tau_s\,h_z(z)\,dz$$

Such that the total impact of the reform on the government objective is

[24] We use
$$h(s,z)=h(s|z)\,h_z(z)$$

[25] We have
$$dz=\frac{\partial z}{\partial\mathcal{T}_z'}\delta\mathcal{T}_z'+\frac{\partial z}{\partial\mathcal{T}_z}\delta\mathcal{T}_z=-\frac{z}{1-\mathcal{T}_z'}\zeta_z^c(z)\,\delta\mathcal{T}_z'-\frac{\eta_z(z)}{1-\mathcal{T}_z'}\delta\mathcal{T}_z$$

[26] We have
$$ds=\frac{\partial s}{\partial T_s'}\delta T_s'+\frac{\partial s}{\partial T_s}\delta T_s+\frac{ds(z;\theta)}{dz}dz=-\frac{s}{1+T_s'}\zeta_{s|z}^c\,\delta T_s'-\frac{\eta_{s|z}(z)}{1+T_s'}\delta T_s+s_{inc}'(z)dz$$

$$
\begin{aligned}
\frac{d\mathcal{L}}{\delta s\delta\tau_s} &= \int_{z\geq z^*}\left\{E_s\left[(1-g(s,z))\,s\middle|z\right]\right\}h_z(z)\,dz &(125)\\
&-\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s\middle|z=z^*\right]h_z(z^*)\\
&-\int_{z\geq z^*}\left\{\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s}{1-T_z'(z)-\tau_s'(z)\,s}\Big(z\zeta_z^c(s,z)s_{inc}'(s,z)+s\,\eta_z(s,z)\Big)\middle|z\right]\right\}h_z(z)\,dz\\
&-\mathbb{E}\left[\frac{s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s\middle|z=z^*\right]\delta z\delta\tau_s\,h_z(z^*)\\
&-\int_{z\geq z^*}\left\{\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c(s,z)\middle|z\right]+\mathbb{E}\left[\frac{s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s_{inc}'(s,z)\middle|z\right]\right\}h_z(z)\,dz &(126)\\
&-\int_{z\geq z^*}\left\{\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\eta_{s|z}(s,z)\middle|z\right]+\mathbb{E}\left[\frac{s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}s\eta_z(s,z)\middle|z\right]\right\}h_z(z)\,dz
\end{aligned}
$$

Redefining augmented social marginal welfare weights as

$$
\hat{g}(s,z)=g(s,z)+\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}\eta_z(s,z)+\frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(s,z) \tag{127}
$$

we finally get

$$
\begin{aligned}
\frac{d\mathcal{L}}{\delta s\delta\tau_s} &= \int_{z\geq z^*}\mathbb{E}\left[(1-\hat{g}(s,z))\,s\middle|z\right]h_z(z)\,dz &(128)\\
&-\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s\middle|z=z^*\right]h_z(z^*)\\
&-\int_{z\geq z^*}\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s_{inc}'(s,z)\Big)\middle|z\right]h_z(z)\,dz\\
&-\int_{z\geq z^*}\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c(s,z)\middle|z\right]h_z(z)\,dz
\end{aligned}
$$

The optimal linear earnings-dependent savings tax rate $\tau_s(.)$ satisfies at each earnings z^*,

$$
\begin{aligned}
&\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s\middle|z=z^*\right]h_z(z^*)+\int_{z\geq z^*}\mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)}s\zeta_{s|z}^c(s,z)\middle|z\right]h_z(z)\,dz\\
&=\int_{z\geq z^*}\mathbb{E}\left[(1-\hat{g}(s,z))\,s\middle|z\right]h_z(z)\,dz-\int_{z\geq z^*}\mathbb{E}\left[\frac{T_z'(z)+\tau_s'(z)\,s+s_{inc}'(s,z)\,\tau_s(z)}{1-T_z'(z)-\tau_s'(z)\,s}z\zeta_z^c(s,z)s_{inc}'(s,z)\middle|z\right]h_z(z)\,dz
\end{aligned} \tag{129}
$$

B.8 Proof of Proposition 6 (bequest taxation and behavioral agents)

We here provide a sufficient statistics characterization of a smooth tax system $\mathcal{T}(s,z)$ under the following additively separable representation of agents' preferences

$$U\big(c,s,z;\theta\big) = u(c;\theta) - k(z;\theta) + \beta(\theta)\,v(s;\theta),$$

and for a utilitarian government that maximizes

$$\int_\theta \big[U(c(\theta),s(\theta),z(\theta);\theta) + \nu(\theta)\,v(s(\theta);\theta)\big]dF(\theta), \tag{130}$$

where $\nu(\theta)$ parametrizes the degree of misalignment on the valuation of the future (generation).

Using the mapping between types θ and earnings z, the Lagrangian of the problem writes

$$\mathcal{L} = \int_z \Big[U(c(z),s(z),z;\theta(z)) + \nu(z)\,v(s(z);\theta(z)) + \lambda(\mathcal{T}(s,z) - E)\Big]dH_z(z), \tag{131}$$

As before, we derive optimal tax formulas by considering reforms of marginal earnings tax rates and marginal savings (or bequests) tax rates. Thanks to the additively separable representation of preferences, there are no income effects on labor supply choices. As a result, the only substantial change is that savings changes now lead to changes in social welfare proportional to the degree of misalignment.

Optimal earnings taxes. A small reform at earnings $z^* = z(\theta^*)$ that consists in a small increase $d\tau_z$ of the marginal earnings tax rate in a small bandwidth dz has the following effect

$$\begin{aligned}\frac{1}{\lambda}\frac{d\mathcal{L}}{d\tau_z dz} &= \int_{z^*}^{\bar{z}} (1-\hat{g}(z))\;dH_z(z)\\ &\quad - \left(\mathcal{T}_z'(s(z^*),z^*) + s_{inc}'(z^*)\left(\mathcal{T}_s'(s(z^*),z^*) + \nu(z^*)\frac{v'(s(z^*))}{\lambda}\right)\right)\frac{z^*}{1-\mathcal{T}_z'(s(z^*),z^*)}\zeta_z^c(z^*)\,h_z(z^*).\end{aligned}$$

In this context, social marginal welfare weights augmented with income effects $\hat{g}(z)$ are equal to

$$\hat{g}(z) = \frac{u'(c(z))}{\lambda} + \left(\mathcal{T}_s'(s(z),z) + \nu(z)\frac{v'(s(z))}{\lambda}\right)\frac{\eta_{s|z}(z)}{1+\mathcal{T}_s'(s(z),z)}$$

and we can use agents' first-order condition for savings, $(1+\mathcal{T}_s')\,u'(c) = \beta v'(s)$, to express the misalignment wedge in terms of the social marginal welfare weights $g(z) := \frac{u'(c(z))}{\lambda}$ as

$$\nu(z)\frac{v'(s(z))}{\lambda} = \frac{\nu(z)}{\beta(z)}g(z)\left(1+\mathcal{T}_s'\right).$$

The optimal schedule of marginal earnings tax rates is thus characterized by

$$\frac{\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)}{1-\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)} = \frac{1}{\zeta_z^c(z^*)}\frac{1}{z^*h_z(z^*)}\int_{z=z^*}^{\bar{z}}\left(1-\hat{g}(z)\right)dH_z(z)$$
$$-s_{inc}'(z^*)\frac{\mathcal{T}_s'\left(s(z^*),z^*\right)}{1-\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)} - s_{inc}'(z^*)\frac{\nu\left(z\right)}{\beta\left(z\right)}g(z)\frac{1+\mathcal{T}_s'\left(s(z^*),z^*\right)}{1-\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)}$$

Optimal savings (or bequests) taxes. A small reform of the savings tax at savings $s^*=s(\theta^*)$ that consists in a small increase $d\tau_s$ of the marginal savings tax rate in a small bandwidth ds has the following effect

$$\frac{1}{\lambda}\frac{d\mathcal{L}}{d\tau_s ds} = s'\left(z^*\right)\int_{z^*}^{\bar{z}}\left(1-\hat{g}(z)\right)dH_z(z)$$
$$-\left\{\left(\mathcal{T}_s'\left(s(z^*),z^*\right)+\nu\left(z^*\right)\frac{v'\left(s\left(z^*\right)\right)}{\lambda}\right)\frac{s(z^*)}{1+\mathcal{T}_s'\left(s\left(z^*\right),z^*\right)}\zeta_{s|z}^c(z^*)+\right.$$
$$\left.\left[\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)+s_{inc}'(z^*)\left(\mathcal{T}_s'\left(s(z^*),z^*\right)+\nu\left(z^*\right)\frac{v'\left(s\left(z^*\right)\right)}{\lambda}\right)\right]\frac{z^*}{1-\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)}\zeta_z^c(z^*)s_{inc}'(z^*)\right\}h_z(z^*)$$

Replacing the misalignment wedge by its expression in terms of social marginal welfare weights $g\left(z\right)$, we obtain that the optimal schedule of marginal savings (or bequest) tax rates is thus characterized by

$$\frac{\mathcal{T}_s'\left(s(z^*),z^*\right)}{1+\mathcal{T}_s'\left(s\left(z^*\right),z^*\right)}\zeta_{s|z}^c(z^*)s(z^*)h_z(z^*)$$
$$= s'\left(z^*\right)\int_{z^*}^{\bar{z}}\left(1-\hat{g}(z)\right)dH_z(z) - s_{inc}'(z^*)\frac{\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)+s_{inc}'(z^*)\mathcal{T}_s'\left(s(z^*),z^*\right)}{1-\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)}\zeta_z^c(z^*)z^*h_z(z^*) \tag{132}$$
$$-\frac{\nu\left(z\right)}{\beta\left(z\right)}g(z)\left[\zeta_{s|z}^c(z^*)s(z^*)+\left(s_{inc}'(z^*)\right)^2\frac{1+\mathcal{T}_s'\left(s(z^*),z^*\right)}{1-\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)}\zeta_z^c(z^*)z^*\right]h_z(z^*)$$

Plugging in the formula for optimal marginal earnings tax rates, we finally obtain

$$\frac{\mathcal{T}_s'\left(s(z^*),z^*\right)}{1+\mathcal{T}_s'\left(s\left(z^*\right),z^*\right)} = s_{pref}'\left(z^*\right)\frac{1}{\zeta_{s|z}^c(z^*)}\frac{1}{s(z^*)h_z(z^*)}\int_{z^*}^{\bar{z}}\left(1-\hat{g}(z)\right)dH_z(z) - \frac{\nu\left(z\right)}{\beta\left(z\right)}g(z)$$

Pareto-efficiency condition. Combining optimal formulas for earnings and savings (or bequests) taxes to eliminate the integral term yields the following Pareto-efficiency condition

$$\frac{\mathcal{T}_s'\left(s(z^*),z^*\right)}{1+\mathcal{T}_s'\left(s\left(z^*\right),z^*\right)}+\frac{\nu\left(z\right)}{\beta\left(z\right)}g(z)$$
$$= s_{pref}'\left(z^*\right)\frac{\zeta_z^c(z^*)z^*}{\zeta_{s|z}^c(z^*)s(z^*)}\left[\frac{\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)+s_{inc}'(z^*)\mathcal{T}_s'\left(s(z^*),z^*\right)}{1-\mathcal{T}_z'\left(s\left(z^*\right),z^*\right)}+s_{inc}'(z^*)\frac{\nu\left(z\right)}{\beta\left(z\right)}g(z)\right].$$

B.9 Proof of Proposition 9 and 10 (implementability with simple tax systems)

B.9.1 Proof of Proposition 9

SN tax system. The sufficient conditions for local optimality under the candidate SN tax system follow directly from Lemma 3 which computes agents' SOCs at the optimal incentive-compatible allocation under a general tax system $\mathcal{T}(s,z)$. Indeed, agents' SOCs are satisfied if Equations (47), (48), and (49) are negative under the SN tax system. Since the cross-partial derivative $\mathcal{T}''_{sz}$ is zero for a SN tax system, it is easy to verify that conditions (39) and (40) on the derivatives of $\mathcal{S}$ and $\mathcal{Z}$, combined with monotonicity ($s^{*\prime}(\theta) > 0$, $s^{*\prime}(z) > 0$) and Assumption 1 on the derivatives of U, jointly imply that each of these three equations is the sum of negative terms. This implies that agents' SOCs are satisfied at the optimal incentive-compatible allocation under the candidate SN tax system.

LED tax system. To derive sufficient conditions for local optimality under the candidate LED tax system, we begin from results obtained in the derivations of Lemma 3 which computes agents' SOCs at the optimal incentive-compatible allocation. We consider the requirements $V''_{ss} < 0$, $V''_{zz} < 0$, and $V''_{ss}V''_{zz} > (V''_{sz})^2$ in turn.

First, to show that V''_{ss} is negative, note that under a LED tax system, $\mathcal{T}''_{ss} = 0$. Therefore, using the fact that under the candidate LED tax system we have $1 + \mathcal{T}'_s = \frac{U'_s}{U'_c}$ at the optimal incentive-compatible allocation, the general expression for V''_{ss} given in Equation (50) reduces to

$$V''_{ss}(s^*(\theta), z^*(\theta); \theta) = \left(\frac{U'_s}{U'_c}\right)^2 U''_{cc} - 2\frac{U'_s}{U'_c}U''_{cs} + U''_{ss}$$

Therefore when utility is additively separable in c and s (implying $U''_{cs} = 0$), the concavity of preferences ($U''_{cc} \leq 0$ and $U''_{ss} \leq 0$) guarantees that this expression is negative.

Second, to show that V''_{zz} is negative, note that under the candidate LED tax system defined in Equations (37) and (38) we have

$$\mathcal{T}''_{sz}(s,z) = \tau'_s(z).$$

We can thus find an expression for $\tau'_s(z)$ at any point in the allocation in question by totally differentiating Equation (37) with respect to θ:

$$\begin{aligned}
\tau'_s\left(z^*(\theta)\right) z^{*\prime}(\theta) &= \frac{d}{d\theta}\Big[\mathcal{S}\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right)\Big] \\
&= \frac{d}{d\theta}\Big[\mathcal{S}\left(z^*(\theta) - s^*(\theta) - \mathcal{T}\left(s^*(\theta), z^*(\theta)\right), s^*(\theta), z^*(\theta); \theta\right)\Big] \\
&= \mathcal{S}'_c \cdot \left[(1-\mathcal{T}'_z)z^{*\prime}(\theta) - \left(1+\mathcal{T}'_s\right)s^{*\prime}(\theta)\right] + \mathcal{S}'_s s^{*\prime}(\theta) + \mathcal{S}'_z z^{*\prime}(\theta) + \mathcal{S}'_\theta,
\end{aligned}$$

which yields

$$\tau_s'\left(z^*(\theta)\right) = \mathcal{S}_c' \cdot \left(1 - \mathcal{T}_z'\right) - \mathcal{S}_c' \cdot \left(1 + \mathcal{T}_s'\right) s^{*\prime}(z^*) + \mathcal{S}_s' \cdot s^{*\prime}(z^*) + \mathcal{S}_z' + \frac{\mathcal{S}_\theta'}{z^{*\prime}(\theta)}.$$

Substituting this into the expression for V_{zz}'' in (57), we have

$$\begin{aligned} V_{zz}''\left(s^*\left(\theta\right), z^*\left(\theta\right); \theta\right) = {} & U_s' s^{*\prime}(z^*)\mathcal{Z}_c' - U_c' s^{*\prime}(z^*)\mathcal{Z}_s' - U_c' \frac{\mathcal{Z}_\theta'}{z^{*\prime}(\theta)} \\ & + U_c' s^{*\prime}(z^*)\left[\mathcal{S}_c' \cdot \left(1 - \mathcal{T}_z'\right) - \mathcal{S}_c' \cdot \left(1 + \mathcal{T}_s'\right) s^{*\prime}(z^*) + \mathcal{S}_s' \cdot s^{*\prime}(z^*) + \mathcal{S}_z' + \frac{\mathcal{S}_\theta'}{z^{*\prime}(\theta)}\right]. \end{aligned} \tag{133}$$

Now employing the assumption that utility is separable in c, s, and z, (implying both $U_{cz}'' = 0$ and $U_{cs}'' = 0$) we have

$$\begin{aligned} U_s'\mathcal{Z}_c' + U_c'\mathcal{S}_c'(1 - \mathcal{T}_z') &= U_s'\mathcal{Z}_c' - U_z'\mathcal{S}_c' \\ &= U_s' \frac{U_c'U_{cz}'' - U_z'U_{cc}''}{\left(U_c'\right)^2} - U_z' \frac{U_c'U_{cs}'' - U_s'U_{cc}''}{\left(U_c'\right)^2} \\ &= 0. \end{aligned}$$

Substituting this result into Equation (133), and noting that $\mathcal{Z}_s' = \mathcal{S}_z' = 0$ by separability, yields

$$V_{zz}''(s^*(\theta), z^*(\theta); \theta) = \left(s^{*\prime}(z^*)\right)^2 \left[U_c'\mathcal{S}_s' - U_s'\mathcal{S}_c'\cdot\right] - \frac{U_c'}{z^{*\prime}(\theta)}\left[\mathcal{Z}_\theta' - s^{*\prime}(z^*)\mathcal{S}_\theta'\right] \tag{134}$$

Again employing separability, we have

$$U_c'\mathcal{S}_s' - U_s'\mathcal{S}_c' = U_c' \frac{U_c'U_{ss}'' - U_s'U_{cs}''}{\left(U_c'\right)^2} - U_s' \frac{U_c'U_{cs}'' - U_s'U_{cc}''}{\left(U_c'\right)^2} = U_{ss}'' + \left(\frac{U_s'}{U_c'}\right)^2 U_{cc}'' \leq 0,$$

implying that the first term on the right-hand side of Equation (134) is negative. The condition $\mathcal{Z}_\theta' - s^{*\prime}(z^*)\mathcal{S}_\theta' \geq 0$ from (41) in the Proposition then implies Equation (134) (and thus V_{zz}'') is negative.

Third, to show $V_{ss}''V_{zz}'' > (V_{sz}'')^2$, we proceed from Equation (49) in Lemma 3:

$$\begin{aligned} & \left(V_{sz}''\right)^2 - V_{ss}''V_{zz}'' \\ & = \frac{U_c'}{s^{*\prime}(\theta)}\left[\left(U_z'\mathcal{S}_c' - U_c'\mathcal{S}_z'\right)\mathcal{Z}_\theta' + \left(U_s'\mathcal{Z}_c' - U_c'\mathcal{Z}_s' - U_c'\frac{\mathcal{Z}_\theta'}{s^{*\prime}(\theta)}\right) s^{*\prime}(z^*)\mathcal{S}_\theta' + \left(\mathcal{Z}_\theta' + s^{*\prime}(z^*)\mathcal{S}_\theta'\right) U_c'\mathcal{T}_{sz}''\right] \\ & = \left(U_z'\mathcal{S}_c' - U_c'\mathcal{S}_z'\right)\frac{U_c'}{s^{*\prime}(\theta)}\mathcal{Z}_\theta' + \frac{U_c'}{s^{*\prime}(\theta)}\mathcal{Z}_\theta' U_c'\mathcal{T}_{sz}'' + \frac{U_c'}{s^{*\prime}(\theta)}\mathcal{S}_\theta'\left(U_s's^{*\prime}(z^*)\mathcal{Z}_c' - U_c's^{*\prime}(z^*)\mathcal{Z}_s' - U_c'\frac{\mathcal{Z}_\theta'}{z^{*\prime}(\theta)} + U_c's^{*\prime}(z^*)\mathcal{T}_{sz}''\right). \end{aligned}$$

Recognizing that the last bracket term is exactly the expression for V''_{zz} given in Lemma 3 this gives

$$\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = \left(U'_z\mathcal{S}'_c - U'_c\mathcal{S}'_z\right)\frac{U'_c}{s^{*\prime}(\theta)}\mathcal{Z}'_\theta + \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{Z}'_\theta U'_c\mathcal{T}''_{sz} + \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta V''_{zz}$$

using the previous expression derived for $\mathcal{T}''_{sz} = \tau'_s$, and the fact that separability ensures $\mathcal{S}'_z = 0$, we obtain after simplification

$$\left(V''_{sz}\right)^2 - V''_{ss}V''_{zz} = -\frac{(U'_c)^2}{s^{*\prime}(\theta)z^{*\prime}(\theta)}\mathcal{Z}'_\theta\left[s^{*\prime}(\theta)\left(\mathcal{S}\cdot\mathcal{S}'_c - \mathcal{S}'_s\right) - \mathcal{S}'_\theta\right] + \frac{U'_c}{s^{*\prime}(\theta)}\mathcal{S}'_\theta V''_{zz}.$$

We have already shown that V''_{zz} is negative. Thus the conditions $\mathcal{S}'_\theta \geq 0$ and $\mathcal{S}'_\theta \leq s^{*\prime}(\theta)\left(\mathcal{S}\cdot\mathcal{S}'_c - \mathcal{S}'_s\right)$ from (41) in the Proposition imply that both terms on the right-hand side are negative, implying that all second-order conditions hold.

B.9.2 Proof of Proposition 10

We begin with a more general statement, and then derive Proposition 10 as a corollary. For a fixed type θ, let $c(z,\theta)$ and $s(z,\theta)$ be its preferred consumption and savings choices at earnings z, given the budget constraint induced by $\mathcal{T}(s,z)$

Proposition 11. *Suppose that $\mathcal{A} = \{(c^*(\theta), s^*(\theta), z^*(\theta))\}_\theta$ constitutes a set of local optima for types θ under a smooth tax system $\mathcal{T}$, where $z^*(\theta)$ is increasing. Individuals' local optima correspond to their global optima when*

1. $\mathcal{Z} = \frac{U'_z(c,s,z;\theta)}{U'_c(c,s,z;\theta)}$ *and* $\mathcal{S} = \frac{U'_s(c,s,z;\theta)}{U'_c(c,s,z;\theta)}$ *are strictly increasing in θ for all (c,s,z)*
2. *For any two types θ and θ', we cannot have both*

$$\begin{aligned}&U'_c\Big(c^*(\theta), s^*(\theta), z^*(\theta);\theta\Big)\sigma_c\Big(s^*(\theta), z^*(\theta)\Big) + U'_z\Big(c^*(\theta), s^*(\theta), z;\theta\Big) \\ &< U'_c\Big(c\left(z^*(\theta),\theta'\right), s\left(z^*(\theta),\theta'\right), z^*(\theta);\theta\Big)\sigma_c\Big(s\left(z^*(\theta),\theta'\right), z^*(\theta)\Big) + U'_z\Big(c\left(z^*(\theta),\theta'\right), s\left(z^*(\theta),\theta'\right), z^*(\theta);\theta\Big)\end{aligned} \tag{135}$$

and

$$\begin{aligned}&U'_s\Big(c^*(\theta), s^*(\theta), z^*(\theta);\theta\Big)\sigma_c\Big(s^*(\theta), z^*(\theta)\Big) + U'_z\Big(c^*(\theta), s^*(\theta), z;\theta\Big)\Big) \\ &< U'_s\Big(c\left(z^*(\theta),\theta'\right), s\left(z^*(\theta),\theta'\right), z^*(\theta);\theta\Big)\sigma_s\Big(s\left(z^*(\theta),\theta'\right), z^*(\theta)\Big) + U'_z\Big(c\left(z^*(\theta),\theta'\right), s\left(z^*(\theta),\theta'\right), z^*(\theta);\theta\end{aligned} \tag{136}$$

where $\sigma_c(s,z) := 1 - \mathcal{T}'_z(s,z)$ and $\sigma_s(s,z) := \frac{1-\mathcal{T}'_z(s,z)}{1+\mathcal{T}'_s(s,z)}$.

Condition 1 corresponds to single crossing assumptions for earnings and savings. Condition 2 is a requirement that if type θ preserves its assigned earnings level $z^*(\theta)$, but chooses some other consumption level s (corresponding to a level that some other type θ' would choose if forced to choose earnings level $z^*(\theta)$), then at this alternative consumption bundle agent θ cannot have both higher marginal utility from increasing its savings through one more unit of work *and* increasing its consumption through one more unit of work. Generally, this condition will hold as long as U is sufficiently concave in consumption and savings when type θ chooses earnings level $z^*(\theta)$.

Proof. To prove agents' local optima are global optima, we want to show that for any given agent θ^*, utility decreases when moving from allocation $(c^*(\theta^*), s^*(\theta^*), z^*(\theta^*))$ to allocation $(c(z,\theta^*), s(z,\theta^*), z)$.

The first step is to compute agent θ^* utility change. The envelope theorem applied to savings choices $s(z,\theta^*)$ implies

$$\begin{aligned}&\frac{d}{dz}U\left(c(z,\theta^*), s(z,\theta^*), z;\theta^*\right)\\&= U_c'\left(c(z,\theta^*), s(z,\theta^*), z;\theta^*\right)\sigma_c\left(s(z,\theta^*), z\right) + U_z'\left(c(z,\theta^*), s(z,\theta^*), z;\theta^*\right)\end{aligned}$$

where $\sigma_c\left(s,z\right) = 1 - \mathcal{T}_z'\left(s,z\right)$. Note that, as established by Milgrom and Segal (2002), these equalities hold as long as U is differentiable in z (holding s and c fixed)—differentiability of $c(z,\theta^*)$ or $s(z,\theta^*)$ is actually not required.

Similarly, the envelope theorem applied to consumption choices $c(z,\theta^*)$ implies

$$\begin{aligned}&\frac{d}{dz}U\left(c(z,\theta^*), s(z,\theta^*), z;\theta^*\right)\\&= U_s'\left(c(z,\theta^*), s(z,\theta^*), z;\theta^*\right)\sigma_s\left(s(z,\theta^*), z\right) + U_z'\left(c(z,\theta^*), s(z,\theta^*), z;\theta^*\right)\end{aligned} \tag{137}$$

where $\sigma_s\left(s,z\right) = \frac{1-\mathcal{T}_z'(s,z)}{1+\mathcal{T}_s'(s,z)}$.

Therefore, agent's θ^* utility change when moving from allocation $(c^*(\theta^*), s^*(\theta^*), z^*(\theta^*))$ to allocation $(c(z,\theta^*), s(z,\theta^*), z)$ is

$$\begin{aligned}&U\left(c(z,\theta^*), s(z,\theta^*), z;\theta^*\right) - U\left(c(z^*(\theta^*),\theta^*), s(z^*(\theta^*),\theta^*), z^*(\theta^*);\theta^*\right)\\&= \int_{x=z^*(\theta^*)}^{x=z}\Big[\min\left\{U_c'\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)\sigma_c\left(s(x,\theta^*), x\right), U_s'\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)\sigma_s\left(s(x,\theta^*), x\right)\right\}\\&\qquad + U_z'\left(c(x,\theta^*), s(x,\theta^*), x;\theta^*\right)\Big]dx\end{aligned} \tag{138}$$

where the min operator is introduced without loss of generality given the fact that both terms are equal.

The second step is to show that under our assumptions, agent θ^* utility change (138) is negative. To do so, let θ_x be the type that chooses earnings x. Then, by definition, agent θ_x utility is maximal at earnings x implying both

$$\begin{aligned}U_c'\left(c^*(\theta_x), s^*(\theta_x), x;\theta_x\right)\sigma_c\left(s^*(\theta_x), x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x;\theta_x\right) = 0\\U_s'\left(c^*(\theta_x), s^*(\theta_x), x;\theta_x\right)\sigma_s\left(s^*(\theta_x), x\right) + U_z'\left(c^*(\theta_x), s^*(\theta_x), x;\theta_x\right) = 0\end{aligned}$$

such that

$$\max\left\{U_c'\left(c^*(\theta_x),s^*(\theta_x),x;\theta_x\right)\sigma_c\left(s^*(\theta_x),x\right),U_s'\left(c^*(\theta_x),s^*(\theta_x),x;\theta_x\right)\sigma_s\left(s^*(\theta_x),x\right)\right\} + U_z'\left(c^*(\theta_x),s^*(\theta_x),x;\theta_x\right)=0 \tag{139}$$

Now, by condition 2, we either have[27]

$$\begin{aligned}&U_c'\left(c^*(\theta_x),s^*(\theta_x),x;\theta_x\right)\sigma_c\left(s^*(\theta_x),x\right)+U_z'\left(c^*(\theta_x),s^*(\theta_x),x;\theta_x\right)\\ &\geq U_c'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)\sigma_c\left(s(x,\theta^*),x\right)+U_z'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)\end{aligned}$$

or

$$\begin{aligned}&U_s'\left(c^*(\theta_x),s^*(\theta_x),x;\theta_x\right)\sigma_s\left(s^*(\theta_x),x\right)+U_z'\left(c^*(\theta_x),s^*(\theta_x),x;\theta_x\right)\\ &\geq U_s'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)\sigma_s\left(s(x,\theta^*),x\right)+U_z'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)\end{aligned}$$

implying that

$$\begin{aligned}&\max\left\{U_c'\left(c^*(\theta_x),s^*(\theta_x),x;\theta_x\right)\sigma_c\left(s^*(\theta_x),x\right),U_s'\left(c^*(\theta_x),s^*(\theta_x),x;\theta_x\right)\sigma_s\left(s^*(\theta_x),x\right)\right\}\\ &+U_z'\left(c^*(\theta_x),s^*(\theta_x),x;\theta_x\right)\\ &\geq \min\left\{U_c'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)\sigma_c\left(s(x,\theta^*),x\right),U_s'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)\sigma_s\left(s(x,\theta^*),x\right)\right\}\\ &+U_z'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)\end{aligned} \tag{140}$$

But since the maximum is zero, this minimum has to be negative. Hence, we have either

$$\begin{aligned}&U_c'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)\sigma_c\left(s(x,\theta^*),x\right)+U_z'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)\leq 0\\ \iff &\frac{U_z'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)}{U_c'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)}\leq -\sigma_c\left(s(x,\theta^*),x\right)\end{aligned}$$

or

$$\begin{aligned}&U_s'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)\sigma_s\left(s(x,\theta^*),x\right)+U_z'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)\leq 0\\ \iff &\frac{U_z'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)}{U_s'\left(c(x,\theta^*),s(x,\theta^*),x;\theta_x\right)}\leq -\sigma_c\left(s(x,\theta^*),x\right)\end{aligned}$$

Suppose that $z>z^*(\theta^*)$ such that $x>z^*(\theta^*)$; the case $z<z^*(\theta^*)$ follows identically. For any $x>z^*(\theta^*)$, the monotonicity of the earnings function means that $\theta_x>\theta^*$. Then, by the single-crossing conditions for $\mathcal{Z}=\frac{U_z'}{U_c'}$ and $\mathcal{S}=\frac{U_s'}{U_c'}$, this means that we have either[28]

$$\frac{U_z'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)}{U_c'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)}\leq -\sigma_c\left(s(x,\theta^*),x\right)$$

[27] Not having $\{a<c \quad and \quad b<c\}$ means having $\{a\geq c \quad \text{or} \quad b\geq d\}$ which implies $\max(a,b)\geq\min(c,d)$

[28] Note that having both $\mathcal{Z}$ and $\mathcal{S}$ increasing in θ also implies that $\frac{\mathcal{Z}}{\mathcal{S}}=\frac{U_z'}{U_s'}$ is increasing in θ.

or

$$\frac{U_z'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)}{U_s'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)} \leq -\sigma_c\left(s(x,\theta^*),x\right)$$

implying that for any $x > z^*(\theta^*)$,

$$\begin{aligned} &\min\left\{U_c'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)\sigma_c\left(s(x,\theta^*),x\right), U_s'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right)\sigma_s\left(s(x,\theta^*),x\right)\right\} \\ &+ U_z'\left(c(x,\theta^*),s(x,\theta^*),x;\theta^*\right) \leq 0. \end{aligned} \tag{141}$$

As a result, the right hand-side of Equation (138) is an integral of negative terms, which shows that

$$U\left(c(z,\theta^*),s(z,\theta^*),z;\theta^*\right) - U\left(c^*(\theta^*),s^*(\theta^*),z^*(\theta^*);\theta^*\right) \leq 0 \tag{142}$$

The case with $z < z^*(\theta^*)$ follows identically, proving Proposition 11. □

Proof of Proposition 10

We now derive Proposition 10 as a consequence Proposition 11 by deriving assumptions under which condition 2 is met for SN and LED tax systems.

SN systems First, suppose that $s < s^*(\theta)$, then $c > c^*(\theta)$. Noting that $\sigma_c = 1 - T_z'\left(z^*\left(\theta\right)\right)$ is not a function of s, we can use $U_{cc}'' \leq 0$ and $U_{cs}'' \geq 0$ to obtain

$$U_c'\left(c^*(\theta),s^*(\theta),z^*(\theta);\theta\right)\sigma_c\left(s^*(\theta),z^*(\theta)\right) \geq U_c'\left(c,s,z^*(\theta);\theta\right)\sigma_c\left(s,z^*(\theta)\right).$$

Further relying on the fact that $U_{cz}'' = 0$ and $U_{sz}'' = 0$, we obtain

$$\begin{aligned} &U_c'\left(c^*(\theta),s^*(\theta),z^*(\theta);\theta\right)\sigma_c\left(s^*(\theta),z^*(\theta)\right) + U_z'\left(c^*(\theta),s^*(\theta),z^*(\theta);\theta\right) \\ &\geq U_c'\left(c,s,z^*(\theta);\theta\right)\sigma_c\left(s,z^*(\theta)\right) + U_z'\left(c,s,z^*(\theta);\theta\right). \end{aligned}$$

Conversely, suppose that $s > s^*(\theta)$, then $c < c^*(\theta)$. We have

$$\begin{aligned} &\frac{d}{ds}\left[\frac{U_s'(z - T_z\left(z\right) - s - T_s\left(s\right), s, z^*(\theta);\theta)}{1 + T_s'(s)}\right] \\ &= -U_{cs}'' + \frac{1}{(1 + T_s'(s))}\left[U_{ss}'' - U_s'\frac{T_{ss}''(s)}{1 + T_s'(s)}\right]. \end{aligned}$$

The condition that $\frac{U_{ss}''(c(s,\theta),s,z^*(\theta);\theta)}{U_s'(c(s,\theta),s,z^*(\theta);\theta)} < \frac{T_{ss}''(s)}{1+T_s'(s)}$, together with $U_{cs}'' > 0$, implies that $\frac{U_s'(c(s,\theta),s,z^*(\theta);\theta)}{1+T_s'(s)}$ is decreasing in s and thus that

$$\frac{U_s'(c^*(\theta),s^*(\theta),z^*(\theta);\theta)}{1 + T_s'(s^*(\theta))} \geq \frac{U_s'(c,s,z^*(\theta);\theta)}{1 + T_s'(s)}.$$

Further relying on the fact that $U_{cz}'' = 0$ and $U_{sz}'' = 0$, and that $\mathcal{T}_s' = T_z'\left(z\right)$ is independant of s, we obtain

$$
\begin{aligned}
& U_s'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right) \sigma_s\left(s^*(\theta), z^*(\theta)\right) + U_z'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right) \\
& \geq U_s'\left(c, s, z^*(\theta); \theta\right) \sigma_s\left(s, z^*(\theta)\right) + U_z'\left(c, s, z^*(\theta); \theta\right).
\end{aligned}
$$

LED systems First, consider a type θ' choosing earnings $z = z^*(\theta) > z^*(\theta')$. We have

$$
\begin{aligned}
& \frac{d}{ds}\left[U_c'\left(z - s - \tau_s\left(z^*(\theta)\right)s - T_z\left(z^*(\theta)\right), s, z^*(\theta); \theta\right)\left(1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s\right)\right] \\
& = U_{cs}''\left(1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s\right) - U_{cc}''\left(1 + \tau_s(z^*(\theta))\right)\left(1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s\right) - U_c'\tau_s'\left(z^*(\theta)\right).
\end{aligned}
$$

The first term is negative because $U_{cs}'' \geq 0$ and $1 - \mathcal{T}_z' = -\mathcal{Z} \geq 0$. Now, the condition that $\mathcal{S} = U_s'/U_c'$ is increasing in θ ensures that a type θ' choosing earnings $z^*(\theta) > z^*(\theta')$ has a desired savings level $s(z^*(\theta), \theta') < s^*(\theta)$. In this case, condition *(a)* of the proposition implies that the remaining terms are negative such that

$$
U_c'\left(z - s - \tau_s(z^*(\theta))s - T(z^*(\theta)), s, z^*(\theta); \theta\right) \sigma_c\left(s, z^*(\theta)\right)
$$

is increasing in s for $s < s^*(\theta)$, where $\sigma_c\left(s, z^*(\theta)\right) = 1 - T_z'(z) - \tau_s'(z)s$. As a result,

$$
\begin{aligned}
& U_c'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right) \sigma_c\left(s^*(\theta), z^*(\theta)\right) \\
& \geq U_c'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right) \sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right)
\end{aligned}
$$

and thus relying on the fact that $U_{cz}'' = 0$ and $U_{sz}'' = 0$, we have

$$
\begin{aligned}
& U_c'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right) \sigma_c\left(s^*(\theta), z^*(\theta)\right) + U_z'\left(c^*(\theta), s^*(\theta), z^*(\theta); \theta\right) \\
& \geq U_c'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right) \sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right) + U_z'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right).
\end{aligned}
$$

Second consider a type θ' choosing $z = z^*(\theta) < z^*(\theta')$. We have

$$
\begin{aligned}
& \frac{d}{ds}\left[U_s'\left(z - s - \tau_s(z^*(\theta))s - T(z^*(\theta)), s, z^*(\theta); \theta\right) \frac{1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s}{1 + \tau_s(z)}\right] \\
& = -U_{cs}''\left(1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s\right) + U_{ss}''\frac{1 - T_z'\left(z^*(\theta)\right) - \tau_s'\left(z^*(\theta)\right)s}{1 + \tau_s(z)} + U_s'\frac{\tau_s'\left(z^*(\theta)\right)}{1 + \tau_s(z)}
\end{aligned}
$$

The first term is negative because $U_{cs}'' \geq 0$ and $1 - \mathcal{T}_z' = -\mathcal{Z} \geq 0$. Now, the condition that $\mathcal{S} = U_s'/U_c'$ is increasing in θ ensures that a type θ' choosing earnings $z = z^*(\theta) < z^*(\theta')$ has a desired savings level $s(z^*(\theta), \theta') > s^*(\theta)$. Hence, condition *(b)* of the proposition implies that the remaining terms are negative such that

$$
U_s'\left(z - s - \tau_s(z^*(\theta))s - T(z^*(\theta)), s, z^*(\theta); \theta\right) \sigma_s\left(s, z^*(\theta)\right)
$$

is decreasing in s for $s > s^*(z)$, where $\sigma_s(s, z^*(\theta)) = \frac{1-T_z'(z)-\tau_s'(z)s}{1+\tau_s(z)}$. This ensures that

$$\begin{aligned} &U_s'(c^*(\theta), s^*(\theta), z^*(\theta); \theta)\,\sigma_c(s^*(\theta), z^*(\theta)) \\ &\geq U_s'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right)\sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right) \end{aligned}$$

and thus relying on the fact that $U_{cz}'' = 0$ and $U_{sz}'' = 0$, we have

$$\begin{aligned} &U_s'(c^*(\theta), s^*(\theta), z^*(\theta); \theta)\,\sigma_c(s^*(\theta), z^*(\theta)) + U_z'(c^*(\theta), s^*(\theta), z^*(\theta); \theta) \\ &\geq U_s'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right)\sigma_c\left(s(z^*(\theta), \theta'), z^*(\theta)\right) + U_z'\left(c(z^*(\theta), \theta'), s(z^*(\theta), \theta'), z^*(\theta); \theta\right). \end{aligned}$$

C Details on the empirical application

C.1 Details on the calibration

C.1.1 Distribution of earnings and savings

The distribution of earnings and savings is calibrated for the U.S. from the Distributional National Accounts micro-files of Piketty et al. (2018). We use individual measures of pretax income (ptinc) and net personal wealth (hweal) as well as the age category (20 to 44 years old, 45 to 64, and above 65) and household information. Discretizing the income distribution into percentiles by age group, our measure of annualized earnings during the working life z at the n-th percentile is constructed by averaging earnings at the n-th percentile across those aged 20 to 44 and those aged 45 to 64. For married households, we use the average earnings of the couple and assign both members of the couple to the same percentile of income. This can be important for households in which only one member of the household is working. If one member of the household is above 65 years old, we only keep the younger spouse in the sample.

To measure accumulated savings over the working life, our measure of annualized savings s at the n-th percentile of income is constructed as the difference between the median wealth of those aged 45 to 64 and those aged 20 to 44, scaled by the average length of the accumulation period (54.5 − 32 = 22.5). We prefer median values to avoid capturing very large savings accumulation related to large inheritances received during working life. For married households, we take household wealth to be the average wealth of its members. This can be important for couples if assets are legally held by one household member. This yields a monotonic distribution of earnings z and savings $s(z)$, and pins down the cross-sectional variation in savings $s'(z)$.

An issue with the calibration is the existence of negative savings at the bottom of the income distribution. While completely possible, this poses an issue for sufficient statics formulas. Indeed, savings responses to tax increases are given by $-\frac{s(z)}{1+T_s'}\zeta_{s|z}^c(z)$ meaning that agents increase their savings in response to tax increases when they hold negative savings. To remedy this issue, we augment agents' savings by a lump-sum amount of \$6,739 which corresponds to social security benefits. Indeed, we think of social security as an effective in-kind savings grant that each worker receives during their representative working years. This amount is computed as follows:[29] retired workers receive on average \$1,514 per month from social security, meaning $12 * 1,514 = \$18,168$ annually. These benefits are received during an average retirement length of 20 years (life expectancy at 65 years old is just over 20 years), and stem from contributions paid over 40 years from 25 (BLS working age) to 64. Discounting this amount at the risk free real interest rate of 1% over a period of 30 years, which is the distance between the midpoint of working life (age 45) and the midpoint of retirement (age 75) yields an effective in-kind savings grant of $\$18,168 * \frac{20}{40} * \frac{1}{1.01^{30}} = \$6,739$. Reassuringly, our numerical results do not seem to be sensitive to this value.

C.1.2 Measures of s'_{inc}

A key input for our sufficient statistics is the marginal propensity to save out of earned income, $s'_{inc}(z) := \frac{\partial s(z)}{\partial z}\big|_{\theta=\theta(z)}$, which relates changes in the amount of pre-tax (or gross) savings s to changes in the amount of pre-tax earnings z. We here show how we convert available empirical estimates of marginal propensities to consume (or save) into measures of $s'_{inc}(z)$, under the assumption that preferences are weakly separable with respect to the disutility of labor supply.

US estimates from Straub (2018). Straub (2018) estimates for the US the elasticity of working-life consumption with respect to working-life permanent income using PSID data. He uses a post-tax measure of permanent income that includes all taxes and transfers and finds an elasticity of 0.7. In our notations, this means that

$$\frac{\partial c(z)}{\partial (z - T(z))} \frac{z - T(z)}{c(z)} = 0.7.$$

where $T(z)$ denotes total taxes net of transfers at pre-tax earnings z. Recognizing that agents' budget constraint is $c(z) = z - s(z) - T(z)$, this implies

$$\frac{\partial s(z)}{\partial (z - T(z))} = 1 - 0.7 \frac{c(z)}{z - T(z)}$$

Assuming the current US tax system is approximately separable nonlinear, we have $T(z) = T_z(z) + T_s(s(z))$ such that $\partial (z - T(z)) = (1 - T'_z(z) - T'_s(s(z)) s'_{inc}(z)) \partial z$ and we finally obtain

$$\frac{1}{1 - T'_z(z) - T'_s(s(z)) s'_{inc}(z)} s'_{inc}(z) = 1 - 0.7 \frac{c(z)}{z - T(z)}$$
$$\iff s'_{inc}(z) = \frac{\left(1 - 0.7 \frac{c(z)}{z - T(z)}\right)(1 - T'_z(z))}{1 + \left(1 - 0.7 \frac{c(z)}{z - T(z)}\right) T'_s(s(z))}. \tag{143}$$

We can then use our calibrated US tax schedule as well as the earnings and savings profiles to obtain a profile of $s'_{inc}(z)$.

[29] We use the SSA Fact Sheet for this computation.

Norwegian estimates from Fagereng et al. (2019). Fagereng et al. (2019) estimate marginal propensities to consume (MPC) across the earnings distribution using information on lottery prizes linked with administrative data in Norway. They find that agents' consumption peaks during the winning year and gradually reverts to their previous value afterwards. Over a 5-year horizon, they estimate agents consume close to 90% of the tax exempt lottery prize, which translates into MPCs of 0.9.

They find little evidence of variation in MPCs across income levels which implies

$$\frac{\partial c(z)}{\partial (z - T(z))} = 0.9$$

Assuming the current US tax system is approximately separable nonlinear, we have $T(z) = T_z(z) + T_s(s(z))$ such that $\partial(z - T(z)) = (1 - T_z'(z) - T_s'(s(z))\, s_{inc}'(z))\, \partial z$ and we finally obtain

$$\frac{1}{(1 - T_z'(z) - T_s'(s(z))\, s_{inc}'(z))} s_{inc}'(z) = 0.1$$
$$\iff s_{inc}'(z) = \frac{0.1\,(1 - T_z'(z))}{1 + 0.1 T_s'(s(z))}$$

We can then use our calibrated US tax schedule to obtain a profile of $s_{inc}'(z)$, under the key assumption that US households have similar MPCs as Norwegian households.

Own US estimates. Based on survey ...

C.1.3 Savings elasticity

We here show how the elasticity of capital gains realizations of around -0.4 over a ten-year period estimated by Agersnap and Zidar (2020) translates into a savings elasticity $\zeta_{s|z}^c$ around 2.

Assuming these risky assets are held for 10 years and yield on average an 8% pretax rate of return. An asset of \$10,000 yields a gross return of $\$10,000 * (1 + 0.08)^{10} = \$21,589$ and thus a capital gains realization of \$11,589. Starting from a capital gains tax rate of 20%, the associated net-of-tax rate of return is $\frac{\$10{,}000 + (1-0.2) * \$21{,}589}{\$10{,}000} = 1.93$.

Now, consider an increase in the capital gains tax rate to 40%. Noting that the elasticity is $-0.4 = \frac{d \ln CG}{d \ln \tau}$, this implies that realizations post-reform would be equal to $\$11,589 * \exp\Big(0.4 * (\ln(0.4) - \ln(0.2))\Big) = \$8,783$. Post-reform savings would then be equal to $\$8,783 * \frac{1}{(1+0.08)^{10} - 1} = \$7,579$, and the associated net-of-tax rate of return would be $\frac{\$7{,}579 + (1-0.4) * \$8{,}783}{\$7{,}579} = 1.70$.

Putting these numbers together, we can then compute the relevant savings elasticity as $\frac{d \ln s}{d \ln rr} = \frac{\ln(7{,}579) - \ln(10{,}000)}{\ln(1.70) - \ln(1.93)} = 2.16$, which gives the result of a savings elasticty $\zeta_{s|z}^c$ around 2.

C.2 Comparison to Golosov et al. (2013)

In their baseline calibration, Golosov et al. (2013) assume agents preferences are CRRA

$$U = \frac{\alpha\left(w^i\right)}{1+\alpha\left(w^i\right)} \ln c_1^i + \frac{1}{1+\alpha\left(w^i\right)} \ln c_2^i - \frac{1}{\sigma}\left(l^i\right)^{\sigma}$$

where c_1 and c_2 represent first period and second period consumption levels and l is labor supply. The risk aversion parameter is here set to $\gamma = 1$, the isoelastic disulity from labor effort is such that $\sigma = 3$, and the taste parameter is given by

$$\alpha\left(w^i\right) = 1.0526\left(w^i\right)^{-0.0036}$$

meaning that it varies from 1.0433 for individuals in the bottom quintile of the earnings distribution (mean hourly wage of \$12.35, in 1992 dollars) to 1.0406 for individuals in the top quintile of the earnings distribution (mean hourly wage of \$25.39, in 1992 dollars). In other words, this parameter is almost constant with income around an average of $\overline{\alpha} = 1.042$.

To translate this structural calibration into our sufficient statistics, note that the earnings elasticity is given by $\zeta_z^c = \frac{1}{\sigma} = 0.33$. The savings elasticity is given by $\zeta_{s|z}^c = \frac{\alpha(1+\mathcal{T}_s')}{1+\alpha(1+\mathcal{T}_s')}$. Approximating the actual schedule of savings tax rates in the US represented in Figure 1 by a constant marginal savings tax rate of 2% yields $\zeta_{s|z}^c = \frac{1.042*(1+0.02)}{1+1.042*(1+0.02)} = 0.52$.

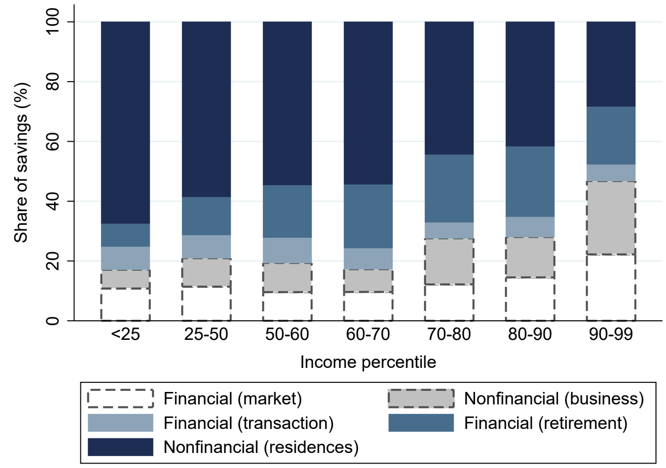
Given the separability assumptions, we can recover the causal income effect on savings s_{inc}' through $s_{inc}' = \frac{1-\mathcal{T}_z'}{1+\mathcal{T}_s'}\eta_{s|z} = \frac{1-\mathcal{T}_z'}{1+\alpha(1+\mathcal{T}_s')}$. Assuming a constant marginal earnings tax rate of 30%, and a constant marginal savings tax rate of 2% yields $s_{inc}' = \frac{1-0.3}{1+1.042*(1+0.02)} = 0.34$.

C.3 Estimates of savings tax rates in the U.S.

Bricker et al. (2019) provide a decomposition of saving types by asset ownership percentile; we summarize the analogous decomposition by income percentile in Figure A1 below. We use the share of savings of each type in Figure A1 to estimate the savings tax rates in Figure 3. We estimate that Financial (market) and Nonfinancial (business) savings are subject to a 15% tax, while the other types of savings are not subject to any additional savings taxes.[30] Thus, we estimate the savings tax rate for each income group as 15%×(share of Financial (market) savings + share of Nonfinancial (business) savings).

[30]The 15% estimated tax on Financial (market) savings and Nonfinancial (business) savings accounts for capital gains taxes. We assume that Financial (transaction) savings—which include checkings and savings accounts among other transaction accounts—and Financial (retirement) savings are tax exempt. We view property taxes on Nonfinancial (residences) savings as a cost on imputed rent, which is paid regardless of whether the asset is owned by the user, so we also assume these savings are tax exempt.

Figure A1: Decomposition of savings types by income—Bricker et al. (2019)



C.4 Inverse optimum approach

The inverse optimum approach aims at inferring the social marginal welfare weights (SMWW) consistent with actual tax policy (Bourguignon and Spadaro, 2012; Lockwood and Weinzierl, 2016). Indeed, assuming the actual policy is optimal, one can invert optimality conditions to obtain the social marginal welfare weights.

In the general case where we consider arbitrary smooth tax systems, we derive two optimality conditions gathered in Proposition 3. Condition (16) characterizes the optimal schedule of marginal earnings tax rates for any given (and potentially suboptimal) schedule of marginal savings tax rates, whereas condition (17) characterizes the optimal schedule of marginal savings tax rates for any given (and potentially suboptimal) schedule of marginal earnings tax rates.

There is thus two methodological choices to be made here. The first choice is whether we assume that the actual schedule of marginal earnings tax rates is optimal, or whether we assume that the actual schedule of marginal savings tax rates is optimal – if both are optimal then the actual tax system should verify Pareto-efficiency which is not the case in the data. The second choice is how to compute the actual tax schedules on earnings and savings, and in particular whether we impose a particular functional form on the actual tax system.

If we assume that the actual schedule of marginal earnings tax rates is optimal, that it only depends on earnings, and that the savings tax is of the separable nonlinear (SN) type, we can infer social marginal welfare weights at each earnings z^* through

$$\frac{T_z'\left(z^*\right)}{1-T_z'\left(z^*\right)} = \frac{1}{\zeta_z^c(z^*)}\frac{1}{z^*h_z(z^*)}\int_{z^*}^{\bar{z}}(1-\hat{g}(x))\,dH_z(x) - s_{inc}'(z^*)\frac{T_s'\left(s\left(z^*\right)\right)}{1-T_z'\left(z^*\right)} \tag{144}$$

$$\iff \int_{z^*}^{\bar{z}}(1-\hat{g}(x))\,dH_z(x) = \zeta_z^c(z^*)z^*h_z(z^*)\frac{T_z'\left(z^*\right)+s_{inc}'(z^*)T_s'\left(s\left(z^*\right)\right)}{1-T_z'\left(z^*\right)} \tag{145}$$

where the right-hand side term can be identified from the data.

Differentiating with respect to z^* yields

$$-\left(1-\hat{g}(z^*)\right)h_z(z^*) = \frac{d\zeta_z^c(z^*)}{dz^*}z^*h_z(z^*)\frac{T_z'\left(z^*\right)+s_{inc}'(z^*)T_s'\left(s\left(z^*\right)\right)}{1-T_z'\left(z^*\right)} + \zeta_z^c(z^*)\left[h_z(z^*)+z^*h_z'(z^*)\right]\frac{T_z'\left(z^*\right)+s_{inc}'(z^*)T_s'\left(s\left(z^*\right)\right)}{1-T_z'\left(z^*\right)} + \zeta_z^c(z^*)z^*h_z(z^*)\frac{d}{dz^*}\left[\frac{T_z'\left(z^*\right)+s_{inc}'(z^*)T_s'\left(s\left(z^*\right)\right)}{1-T_z'\left(z^*\right)}\right] \tag{146}$$

and further assuming that the compensated elasticity of earnings is approximately constant across incomes ($\frac{d\zeta_z^c(z^*)}{dz^*}\approx 0$) yields

$$\hat{g}(z^*) = 1+\zeta_z^c(z^*)\left[1+\frac{z^*h_z'(z^*)}{h_z(z^*)}\right]\frac{T_z'\left(z^*\right)+s_{inc}'(z^*)T_s'\left(s\left(z^*\right)\right)}{1-T_z'\left(z^*\right)} + \zeta_z^c(z^*)z^*\left[\frac{\left[1+s_{inc}'(z^*)T_s'\left(s\left(z^*\right)\right)\right]T_z''\left(z^*\right)}{\left[1-T_z'\left(z^*\right)\right]^2}+\frac{s_{inc}''(z^*)T_s'\left(s\left(z^*\right)\right)+s_{inc}'(z^*)s'(z^*)T_s''\left(s\left(z^*\right)\right)}{1-T_z'\left(z^*\right)}\right] \tag{147}$$

which gives augmented social marginal welfare weights as a function of the local Pareto parameter of the income distribution $1+\frac{z^*h_z'(z^*)}{h_z(z^*)}$ and other observables.

Using the fact that augmented social marginal welfare weights are defined as

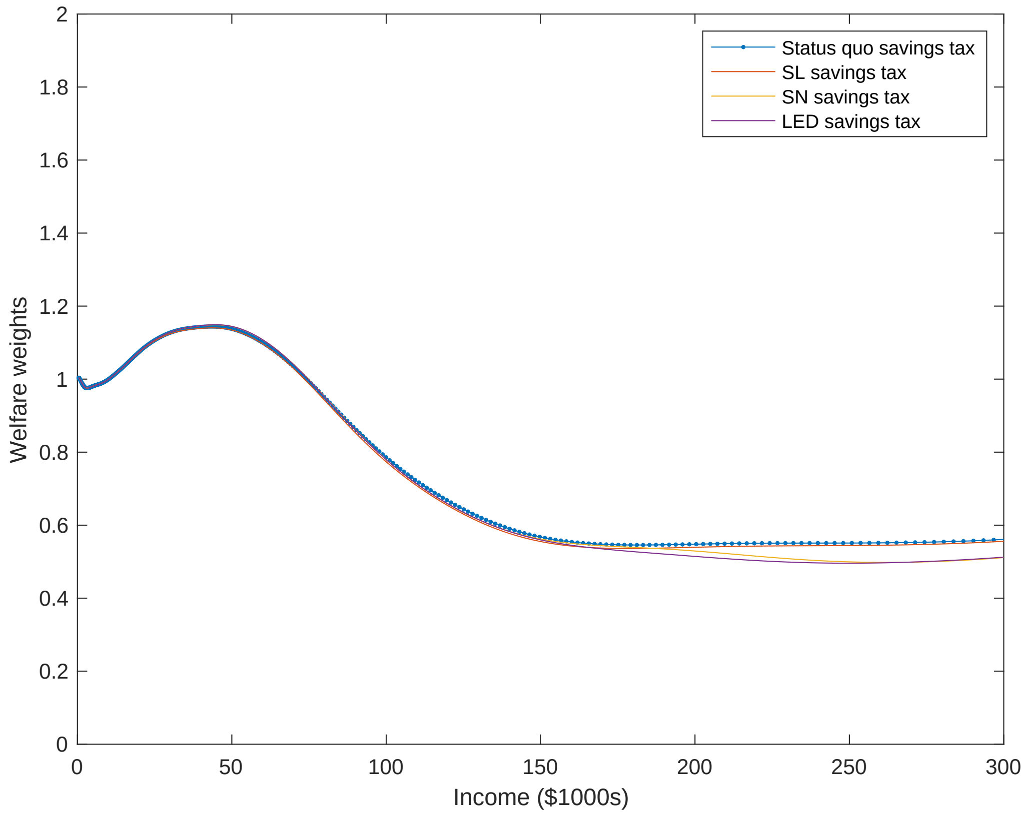
$$\hat{g}(z) := g(z)+T_z'\left(z\right)\frac{\eta_z(z)}{1-T_z'\left(z\right)}+T_s'\left(s(z)\right)\left(\frac{\eta_{s|z}(z)}{1+T_s'\left(s(z)\right)}+s_{inc}'(z)\frac{\eta_z(z)}{1-T_z'\left(z\right)}\right) \tag{148}$$

and assuming agents' preferences are weakly separable, such that by Proposition 2 we have $s_{inc}'\left(z\right) = \frac{1-T_z'(z)}{1+T_s'(s(z))}\eta_{s|z}\left(z\right)$, we finally get

$$g(z^*) = 1+\zeta_z^c(z^*)\left[1+\frac{z^*h_z'(z^*)}{h_z(z^*)}\right]\frac{T_z'\left(z^*\right)+s_{inc}'(z^*)T_s'\left(s\left(z^*\right)\right)}{1-T_z'\left(z^*\right)} + \zeta_z^c(z^*)z^*\left[\frac{\left[1+s_{inc}'(z^*)T_s'\left(s\left(z^*\right)\right)\right]T_z''\left(z^*\right)}{\left[1-T_z'\left(z^*\right)\right]^2}+\frac{s_{inc}''(z^*)T_s'\left(s\left(z^*\right)\right)+s_{inc}'(z^*)s'(z^*)T_s''\left(s\left(z^*\right)\right)}{1-T_z'\left(z^*\right)}\right] - T_z'\left(z\right)\frac{\eta_z(z)}{1-T_z'\left(z\right)}-s_{inc}'\left(z\right)T_s'\left(s(z)\right)\frac{1+\eta_z(z)}{1-T_z'\left(z\right)}. \tag{149}$$

This expression depends on both the current schedule of marginal earnings tax rates, $T_z'(z)$, and savings tax rates, T_s'. We seek the schedule of weights that rationalizes the existing income tax schedule, and thus we set $T_z'(z)$ to reflect the observed schedule of income tax rates. Since the current schedule of savings tax rates is not Pareto efficient, there is a question of whether we compute weights assuming that the savings tax were counterfactually optimal, or whether we use the status quo (suboptimal) savings tax schedule. For consistency with the inverse "optimum" motivation, we use the former approach, but results look essentially identical if we instead use the suboptimal status quo schedule. Figure TK plots our estimated profile of inverse optimum weights, both under the assumption that marginal savings tax rates are SN Pareto-efficient, and assuming the status quo schedule of savings tax rates.

Figure A2: Schedule of Inverse Optimum Social Welfare Weights in the U.S.



Notes: This figure plots the schedule of inverse optimum welfare weights that would rationalize the U.S. income tax schedule. Separate schedules are plotted using the status quo (Pareto-inefficient) savings tax, as well as under the assumption that the savings tax were (counterfactually) Pareto-efficient, with either a separable linear, separable nonlinear, or linear earnings-dependent structure.

C.5 Simulations of optimal savings taxes with multidimensional heterogeneity

C.5.1 Overview and assumptions

Through numerical simulations, we assess how the presence of multidimensional heterogeneity affects optimal savings taxes in each of the simple tax system that we consider. To this end, we use optimal savings tax formulas derived in the presence of multidimensional heterogeneity (Proposition 5).

We assume that the existing earnings tax schedule is optimal to infer the government objective in an inverse optimum approach. This pins down social marginal welfare weights $g(z)$ that are consistent with actual tax policy under unidimensional heterogeneity. Further assuming that social marginal welfare weights are independent of savings s, we can then simulate optimal savings taxes under multidimensional heterogeneity keeping the government objective constant.

Doing so requires a number of assumptions regarding the structure of multidimensional heterogeneity. First, we assume that the compensated elasticity of earnings ζ_z^c and savings $\zeta_{s|z}^c$ are constant across earnings and savings, thereby disregarding any heterogeneity in these parameters.

Second, we assume that earnings income effects can be neglected (i.e. $\eta_z \approx 0$) meaning that changes in tax liabilities do not trigger labor supply responses – this simplifying assumption is by no means crucial for this simulation exercise because it is made for both unidimensional and multidimensional cases.

Third, we assume that agents' preferences are weakly separable, such that by Proposition 2 we have $s'_{inc}(z) = \frac{1-T'_z(z)}{1+T'_s(s(z))}\eta_{s|z}(z)$. Since we identify $s'_{inc}(z)$ from the data, this allows to pin down the income effect parameter on savings $\eta_{s|z}(z)$ in the unidimensional case. We then infer the value of $s'_{inc}(s,z)$ in the multidimensional case by assuming that the income elasticity of savings is constant within earnings and equal to its unidimensional counterpart. Formally, this means that at a given earnings z, for any savings s we have $\eta_{s|z}(s,z) = \frac{s}{\overline{s(z)}}\eta_{s|z}(z)$, where $\overline{s(z)} := \mathbb{E}\left[s\middle|z\right]$ denotes the average savings level at earnings z. And similarly, $s'_{inc}(s,z) = \frac{s}{\overline{s(z)}}s'_{inc}(z)$.

Under these assumptions, we derive for each type of simple tax system an optimal savings tax formula amenable to numerical simulations in the presence of multidimensional heterogeneity.

C.5.2 Separable linear (SL) tax system

The optimal savings tax formula with multidimensional heterogeneity (Proposition 5) is

$$\begin{aligned}&\frac{\tau_s}{1+\tau_s}\int_z \left\{\mathbb{E}\left[s\zeta_{s|z}^c(s,z)\middle|z\right]\right\}dH_z(z) \\ &= \int_z \left\{\mathbb{E}\left[(1-\hat{g}(s,z))\,s\middle|z\right] - \mathbb{E}\left[\frac{T'_z(z)+s'_{inc}(s,z)\,\tau_s}{1-T'_z(z)} z\zeta_z^c(s,z)\,s'_{inc}(s,z)\middle|z\right]\right\}dH_z(z)\end{aligned} \tag{150}$$

Under the aforementioned assumptions, expanding $\hat{g}(s,z)$, replacing $s'_{inc}(s,z)$ and $\eta_{s|z}(s,z)$ by their values, and assuming η_z is negligible gives

$$\begin{aligned}&\frac{\tau_s}{1+\tau_s}\int_z \left\{\overline{s(z)}\zeta_{s|z}^c\right\}dH_z(z) \\ &= \int_z \left\{\mathbb{E}\left[\left(1-g(z)-\tau_s\frac{\eta_{s|z}(z)}{1+\tau_s}\frac{s}{\overline{s(z)}}\right)s\middle|z\right] - \frac{z\zeta_z^c s'_{inc}(z)}{1-T'_z(z)}\mathbb{E}\left[T'_z(z)\frac{s}{\overline{s(z)}} + s'_{inc}(z)\,\tau_s\left(\frac{s}{\overline{s(z)}}\right)^2\middle|z\right]\right\}dH_z(z)\end{aligned} \tag{151}$$

which after rearranging yields

$$\begin{aligned}&\frac{\tau_s}{1+\tau_s}\zeta_{s|z}^c\int_z \overline{s(z)}dH_z(z) \\ &= \int_z \left\{(1-g(z))\,\overline{s(z)} - \frac{\tau_s}{1+\tau_s}\frac{\eta_{s|z}(z)}{\overline{s(z)}}\mathbb{E}\left[s^2\middle|z\right] - \frac{z\zeta_z^c s'_{inc}(z)}{1-T'_z(z)}\mathbb{E}\left[\frac{T'_z(z)}{\overline{s(z)}}s + \frac{s'_{inc}(z)\,\tau_s}{\overline{s(z)}^2}s^2\middle|z\right]\right\}dH_z(z).\end{aligned} \tag{152}$$

We can now use $\mathbb{E}\left[s^2\middle|z\right] = \mathbb{E}\left[\left(s-\overline{s\left(z\right)}\right)^2 + 2s\overline{s\left(z\right)} - \overline{s\left(z\right)}^2\middle|z\right] = \mathbb{V}\left(s|z\right) + \overline{s\left(z\right)}^2$ to obtain

$$\begin{aligned}
&\frac{\tau_s}{1+\tau_s}\zeta^c_{s|z}\int_z \overline{s\left(z\right)}dH_z\left(z\right) \qquad (153)\\
&= \int_z\left\{\left[1-g(z)-\frac{\tau_s\eta_{s|z}\left(z\right)}{1+\tau_s}\left(1+\frac{\mathbb{V}\left(s|z\right)}{\overline{s\left(z\right)}^2}\right)\right]\overline{s\left(z\right)} - \frac{z\zeta^c_z s'_{inc}\left(z\right)}{1-T'_z\left(z\right)}\left[T'_z\left(z\right)+s'_{inc}\left(z\right)\tau_s\left(1+\frac{\mathbb{V}\left(s|z\right)}{\overline{s\left(z\right)}^2}\right)\right]\right\}dH_z\left(z\right)
\end{aligned}$$

which we can finally rewrite as

$$\begin{aligned}
&\frac{\tau_s}{1+\tau_s}\int_z \overline{s\left(z\right)}\zeta^c_{s|z}dH_z\left(z\right) \qquad (154)\\
&= \int_z\left\{\left(1-g(z)-\frac{\tau_s}{1+\tau_s}\eta_{s|z}\left(z\right)\right)\overline{s\left(z\right)} - \frac{T'_z\left(z\right)+s'_{inc}\left(z\right)\tau_s}{1-T'_z\left(z\right)}z\zeta^c_z s'_{inc}\left(z\right)\right\}dH_z\left(z\right)\\
&-\int_z\left\{\underbrace{\frac{\mathbb{V}\left(s|z\right)}{\overline{s\left(z\right)}^2}}_{\geq 0}\underbrace{\left(\frac{\tau_s}{1+\tau_s}\eta_{s|z}\left(z\right)\overline{s\left(z\right)}+\frac{s'_{inc}\left(z\right)\tau_s}{1-T'_z\left(z\right)}z\zeta^c_z s'_{inc}\left(z\right)\right)}_{\geq 0}\right\}dH_z\left(z\right).
\end{aligned}$$

The first two lines correspond to the optimal savings tax formula under unidimensional heterogeneity (Proposition 7) and the last line captures the effect of multidimensional heterogeneity through $\mathbb{V}\left(s|z\right)$. Multidimensional heterogeneity adds a corrective term which is unambiguously negative, it thus prescribes a lower linear savings tax rate.

C.5.3 Separable nonlinear (SN) tax system

The optimal savings tax formula with multidimensional heterogeneity (Proposition 5) is

$$\begin{aligned}
&\frac{T'_s\left(s^*\right)}{1+T'_s\left(s^*\right)}\int_z\left\{s^*\zeta^c_{s|z}(s^*,z)\right\}h\left(s^*,z\right)dz \qquad (155)\\
&= \int_z\left\{\mathbb{E}\left[1-\hat{g}\left(s,z\right)\middle|z,s\geq s^*\right]\right\}h_z\left(z\right)dz - \int_z\left\{\frac{T'_z\left(z\right)+s'_{inc}\left(s^*,z\right)T'_s\left(s^*\right)}{1-T'_z\left(z\right)}z\zeta^c_z(s^*,z)s'_{inc}\left(s^*,z\right)\right\}h\left(s^*,z\right)dz.
\end{aligned}$$

Under the aforementioned assumptions, expanding $\hat{g}\left(s,z\right)$ and assuming η_z is negligible gives

$$\begin{aligned}
&\frac{T'_s\left(s^*\right)}{1+T'_s\left(s^*\right)}s^*\zeta^c_{s|z}\int_z h\left(s^*,z\right)dz \qquad (156)\\
&= \int_{s^*}^{\bar{s}}\left\{\int_z\left[1-g(z)-\frac{T'_s\left(s\right)}{1+T'_s\left(s\right)}s\frac{\eta_{s|z}\left(z\right)}{\overline{s\left(z\right)}}\right]h\left(s,z\right)dz\right\}ds - \int_z\left[\frac{T'_z\left(z\right)+s'_{inc}\left(s^*,z\right)T'_s\left(s^*\right)}{1-T'_z\left(z\right)}z\zeta^c_z s'_{inc}\left(s^*,z\right)\right]h\left(s^*,z\right)d
\end{aligned}$$

or equivalently, expressing this as a function of the savings density $h_s\left(s\right)=\int_z h\left(s,z\right)dz$,

$$\frac{T_s'(s^*)}{1+T_s'(s^*)} s^* \zeta_{s|z}^c h_s(s^*) \tag{157}$$
$$= \int_{s^*}^{\bar{s}} \left\{ \mathbb{E}\left[1 - g(z) - \frac{T_s'(s)}{1+T_s'(s)} \eta_{s|z}(s,z) \Big| s \right] \right\} h_s(s)\, ds - \mathbb{E}\left[\frac{T_z'(z) + s_{inc}'(s,z) T_s'(s)}{1-T_z'(z)} z \zeta_z^c s_{inc}'(s,z) \Big| s = s^* \right] h_s(s^*)$$

where expectations operator denote integration with respect to earnings conditional on savings.

For implementation, we assume that at each point in the income continuum, there are M different equal-sized saver-bins (e.g., bottom-, middle-, and top-third savers), indexed by $m = 1, \ldots, M$. Thus we can write $s_m(z)$ as the savings map for saver-bin m at each income, with $s_m'(z)$ the cross-sectional savings profile within each saver-bin. Then the income density in each saver-bin is $h_{z,m}(z) = h(z)/M$, since the bins are equally sized conditional on income. The savings density among saver-bin m is therefore $h_{s,m}(s) = h_{z,m}(z)/s_m'(z)$, and we have $H(s) = \sum_{m=1}^{M} \int_{s=0}^{\infty} h_{s,m}(s) ds$, and $h_s(s) = \sum_{m=1}^{M} h_{s,m}(s)$. And the savings-conditional average of some $x(s,z)$ is $\mathbb{E}[x(s,z)|s] = \frac{\sum_{m=1}^{M} x(s_m,z) h_{s,m}(s)}{h_s(s)}$.

To better picture the link with the unidimensional formula (30), let also rewrite the latter as a function of the savings density $h_s(s)$ – implicitly defining $z(s)$ as the earnings level of agents with savings s – this yields

$$\frac{T_s'(s^*)}{1+T_s'(s^*)} s^* \zeta_{s|z}^c h_s(s^*) \tag{158}$$
$$= \int_{s^*}^{\bar{s}} \left\{ 1 - g(z(s)) - \frac{T_s'(s)}{1+T_s'(s)} \eta_{s|z}(z(s)) \right\} h_s(s)\, ds - \frac{T_z'(z(s^*)) + s_{inc}'(z(s^*)) T_s'(s^*)}{1-T_z'(z(s^*))} z(s^*) \zeta_z^c s_{inc}'(z(s^*)) h_s(s^*).$$

While it is clear that the multidimensional formula extends the unidimensional formula, determining the impact of multidimensional heterogeneity on tax rates is analytically more difficult and we thus rely on numerical simulations.

C.5.4 Linear earnings dependent (LED) tax system

The optimal savings tax formula in the presence of multidimensional heterogeneity (Proposition 5) is

$$\int_{z \geq z^*} \mathbb{E}\left[\frac{\tau_s(z)}{1+\tau_s(z)} s \zeta_{s|z}^c(s,z) \Big| z \right] dH_z(z) + \mathbb{E}\left[\frac{T_z'(z) + \tau_s'(z) s + s_{inc}'(s,z) \tau_s(z)}{1 - T_z'(z) - \tau_s'(z) s} z \zeta_z^c(s,z) s \Big| z^* \right] h_z(z^*)$$
$$= \int_{z \geq z^*} \left\{ \mathbb{E}\Big[(1 - \hat{g}(s,z)) s \Big| z \Big] - \mathbb{E}\left[\frac{T_z'(z) + \tau_s'(z) s + s_{inc}'(s,z) \tau_s(z)}{1 - T_z'(z) - \tau_s'(z) s} z \zeta_z^c(s,z) s_{inc}'(s,z) \Big) \Big| z \right] \right\} dH_z(z)$$

Under the aforementioned assumptions, expanding $\hat{g}(s,z)$, replacing $s_{inc}'(s,z)$ and $\eta_{s|z}(s,z)$ by their values, and assuming η_z is negligible gives

$$\int_{z\geq z^*}\frac{\tau_s(z)}{1+\tau_s(z)}\overline{s(z)}\zeta^c_{s|z}dH_z(z) \tag{159}$$

$$+z^*\zeta^c_z\mathbb{E}\left[\frac{T'_z(z)+\tau'_s(z)\,s}{1-T'_z(z)-\tau'_s(z)\,s}s+\frac{\tau_s(z)}{1-T'_z(z)-\tau'_s(z)\,s}\frac{s'_{inc}(z)}{\overline{s(z)}}s^2\Big|z^*\right]h_z(z^*)$$

$$=\int_{z\geq z^*}\Bigg\{\mathbb{E}\left[\left(1-g(z)-\frac{\tau_s(z)}{1+\tau_s(z)}\frac{\eta_{s|z}(z)}{\overline{s(z)}}s\right)s\Big|z\right]$$

$$-z\zeta^c_z\frac{s'_{inc}(z)}{\overline{s(z)}}\mathbb{E}\left[\frac{T'_z(z)+\tau'_s(z)\,s}{1-T'_z(z)-\tau'_s(z)\,s}s+\frac{\tau_s(z)}{1-T'_z(z)-\tau'_s(z)\,s}\frac{s'_{inc}(z)}{\overline{s(z)}}s^2\Big)\Big|z\right]\Bigg\}dH_z(z) \tag{160}$$

Since the marginal tax rate on earnings $T'_z(z)+\tau'_s(z)\,s$ features savings s, it is hard to further simplify this formula while retaining an exact characterization. To obtain exact results, we must rely on numerical simulations. We can neverthless derive simpler formulas, either relying on approximations or making additional assumptions.

Using approximations. We can nevertheless provide simple analytical formulas disregarding this dependence and setting $s=\overline{s(z)}$ in marginal earnings tax rates. We believe that these formulas are informative in that they converge to exact expressions as the linear earnings dependent savings tax rate tends to a simple linear savings tax rate – that is $\tau'_s(z)=0$ for all z. Moreover, although these approximations are not unbiased in that they provide an upper bound on the linear-earnings dependent savings tax rate, these upper bounds are tight as the approximation only amounts to assuming $\tau'_s(z^*)\,\mathbb{V}(s|z^*)$ is negligible.[31]

Further using $\mathbb{E}\left[s^2\Big|z^*\right]=\mathbb{V}(s|z^*)+\overline{s(z^*)}^2$, our approximation for the optimal savings tax

[31]Indeed, because functions $X\mapsto\frac{X}{1-X}$ and $X\mapsto\frac{1}{1-X}$ are positive and increasing in X over $[0,1]$ the interval where optimal earnings marginal tax rates lie, we have that

$$\mathbb{E}\left[\frac{T'_z(z)+\tau'_s(z)\,s}{1-T'_z(z)-\tau'_s(z)\,s}s\Big|z^*\right]\geq\frac{T'_z(z^*)+\tau'_s(z^*)\,\overline{s(z^*)}}{1-T'_z(z^*)-\tau'_s(z^*)\,\overline{s(z^*)}}\mathbb{E}\left[s\Big|z^*\right] \tag{161}$$

$$\mathbb{E}\left[\frac{\tau_s(z)}{1-T'_z(z)-\tau'_s(z)\,s}s^2\Big|z^*\right]\geq\frac{\tau_s(z^*)}{1-T'_z(z^*)-\tau'_s(z^*)\,\overline{s(z^*)}}\mathbb{E}\left[s^2\Big|z^*\right] \tag{162}$$

formula is

$$\begin{aligned}
&\int_{z\geq z^*}\frac{\tau_s(z)}{1+\tau_s(z)}\overline{s(z)}\zeta^c_{s|z}dH_z(z) \qquad (163)\\
&+z^*\zeta^c_z\overline{s(z)}\left[\frac{T'_z(z^*)+\tau'_s(z^*)\overline{s(z^*)}}{1-T'_z(z^*)-\tau'_s(z^*)\overline{s(z^*)}}+\frac{s'_{inc}(z)\tau_s(z^*)}{1-T'_z(z^*)-\tau'_s(z^*)\overline{s(z^*)}}\left(1+\frac{\mathbb{V}(s|z^*)}{\overline{s(z^*)}^2}\right)\right]h_z(z^*)\\
&=\int_{z\geq z^*}\left\{(1-g(z))\overline{s(z)}-\frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(z)\overline{s(z)}\left(1+\frac{\mathbb{V}(s|z^*)}{\overline{s(z^*)}^2}\right)\right.\\
&\left.-z\zeta^c_z s'_{inc}(z)\left[\frac{T'_z(z^*)+\tau'_s(z^*)\overline{s(z^*)}}{1-T'_z(z^*)-\tau'_s(z^*)\overline{s(z^*)}}+\frac{s'_{inc}(z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s}\left(1+\frac{\mathbb{V}(s|z^*)}{\overline{s(z^*)}^2}\right)\right]\right\}dH_z(z)
\end{aligned}$$

such that finally we obtain

$$\begin{aligned}
&\int_{z^*}^{\bar z}\left\{\frac{\tau_s(z)}{1+\tau_s(z)}\overline{s(z)}\zeta^c_{s|z}\right\}dH_z(z)+\left[\frac{T'_z(z^*)+\tau'_s(z^*)\overline{s(z^*)}}{1-T'_z(z^*)-\tau'_s(z^*)\overline{s(z^*)}}+\frac{s'_{inc}(z)\tau_s(z^*)}{1-T'_z(z^*)-\tau'_s(z^*)\overline{s(z^*)}}\right]z^*\zeta^c_z\overline{s(z)}h_z(z^*)\\
&=\int_{z^*}^{\bar z}\left\{(1-g(z))\overline{s(z)}-\frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(z)\overline{s(z)}-\left[\frac{T'_z(z)+\tau'_s(z)\overline{s(z)}+s'_{inc}(z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)\overline{s(z)}}\right]z\zeta^c_z s'_{inc}(z)\right\}dH_z(z)\\
&-\underbrace{\frac{\mathbb{V}(s|z^*)}{\overline{s(z^*)}^2}}_{\geq 0}\underbrace{\frac{s'_{inc}(z^*)\tau_s(z^*)}{1-T'_z(z^*)-\tau'_s(z^*)\overline{s(z^*)}}z^*\zeta^c_z\overline{s(z^*)}h_z(z^*)}_{\geq 0} \qquad (164)\\
&-\int_{z^*}^{\bar z}\underbrace{\frac{\mathbb{V}(s|z)}{\overline{s(z)}^2}}_{\geq 0}\underbrace{\left\{\frac{\tau_s(z)}{1+\tau_s(z)}\eta_{s|z}(z)\overline{s(z)}+\frac{s'_{inc}(z)\tau_s(z)}{1-T'_z(z)-\tau'_s(z)s}z\zeta^c_z s'_{inc}(z)\right\}}_{\geq 0}dH_z(z)
\end{aligned}$$

The first two lines correspond to the optimal savings tax formula under unidimensional heterogeneity (Proposition 7), while the third and fourth lines capture the effect of multidimensional heterogeneity through $\mathbb{V}(s|z)$. Multidimensional heterogeneity adds corrective terms which are unambiguously positive, it thus prescribes lower linear earnings-dependent savings tax rates.

D Details of tax systems by country

In Table 2, we consider five categories of savings subject to various taxation regimes in different countries: (i) wealth, (ii) capital gains, (iii) property, (iv) pensions, and (v) inheritance, which are typically defined in tax codes as follows. First, wealth, which is free from taxation in most advanced economies, is defined as the aggregate value of certain classes of assets, such as real estate, stocks, and bank deposits. Next, capital gains consist of realized gains from financial and real estate investments, and include interest and dividend payments. Third, property consists of real estate holdings, such as land, private residences, and commercial properties. Fourth, for our purposes, pensions are defined as private retirement savings in dedicated accounts, excluding government transfers to retired individuals, such as Social Security in the United States. Lastly, inheritances—also known as estates—are the collections of assets bequeathed by deceased individuals to living individuals, often relatives.

For each country, we label the tax system applied to each category of savings with the types described in Table 1 or "Other," which encompasses all other tax systems. An additional common simple tax structure is a "composite" tax, in which savings and labor income are not distinguished for the purposes of taxation. Composite taxes are often applied to classes of income for which it is unclear whether the income should be considered capital income or labor income. For example, in a majority of the countries in Table 2, rental income—which requires some active participation from the recipient of the income—is subject to composite taxation.

In the subsections below, we have included additional details about the tax system in each country in Table 2. Note that we characterize tax systems that feature a flat tax on savings above an exempt amount as having a separable nonlinear tax system. In addition, when benefits are withdrawn from pension accounts, they are often subject to the same progressive tax rates as labor income. We characterize these tax systems as separable nonlinear rather than composite since benefits are generally received after retirement from the labor force when the taxpayer's income is primarily composed of savings.

Australia

- **Wealth:** No wealth tax.
- **Capital gains:** Generally a composite tax applies. Gains from certain assets are exempt or discounted.
- **Property:** At the state level, land tax rates are progressive; primary residence land is typically exempt. At the local level, generally flat taxes are assessed on property but the taxes can be nonlinear as well, depending on the locality.
- **Pensions:** A flat tax is assessed on capital gains made within the pension account. A component of pension benefits may be subject to taxation when withdrawn, in which case the lesser of a flax tax or the same progressive tax rates as apply to labor income is assessed.
- **Inheritance:** No inheritance tax.

Austria

- **Wealth:** No wealth tax.
- **Capital gains:** Generally a flat tax is assessed, with the rate depending on the type of asse; taxpayers with lower labor income can opt to apply their labor income tax rate instead. Gains from certain classes of assets are exempt.
- **Property:** Either flat or progressive tax rates are assessed on property, depending on its intended use. Rates vary by municipality.

- **Pensions:** Generally no tax on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income, with discounts applicable to certain types of withdrawals.
- **Inheritance:** No inheritance tax.

Canada

- **Wealth:** No wealth tax.
- **Capital gains:** For most capital gains, a discount is first applied to the gain and then the discounted gain is added to labor income and taxed progressively. For certain gains, such as interest income, no discount is applied. Lifetime exemptions up to a limit apply to gains from certain classes of assets.
- **Property:** Generally a flat tax is assessed on property, with rates varying by province and locality.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income, with exemptions applicable to certain types of withdrawals.
- **Inheritance:** No separate inheritance tax. A final year tax return is prepared for the deceased, including income for that year, that treats all assets as if they have just been sold and applies the relevant taxes (e.g., labor income and capital gains taxes) accordingly.

Denmark

- **Wealth:** No wealth tax.
- **Capital gains:** Progressive taxation with two tax brackets. Gains from certain classes of assets are exempt.
- **Property:** At the national level, property is subject to progressive taxation with two tax brackets. Pensioners under an income threshold can receive tax relief. Land taxes—assessed at the local level—are flat taxes, with rates varying by municipality.
- **Pensions:** A flat tax is assessed on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income (excluding a labor market surtax), a flat tax, or are exempt from taxation, depending on the type of pension.
- **Inheritance:** Generally a flat tax is assessed on the inheritance above an exemption, with a higher tax rate for more distant relatives. Transfers to spouses and charities are exempt. Inheritances above a certain value are subject to additional taxes.

France

- **Wealth:** No wealth tax.
- **Capital gains:** Different rates—progressive and flat—apply to gains from different classes of assets. Certain low-income individuals are either exempt from taxes or can opt to apply their labor income tax rate, depending on the type of asset. High-income individuals are subject to a surtax. Gains from certain assets are exempt or discounted.
- **Property:** Residence taxes are assessed on property users, while property taxes on developed and undeveloped properties are assessed on owners. Rates are set at the local level and apply to the estimated rental value of the property. Exemptions, reductions, and surcharges may apply depending on the taxpayer's reference income and household composition, certain events, and property characteristics. Surcharges may also apply to higher-value properties. An additional property wealth tax applies at the national level; rates are progressive above an exemption.
- **Pensions:** Generally no tax on capital gains made within the pension account. Pension benefits beyond an exemption are generally subject to the same progressive tax rates as labor income. A flat tax is assessed on certain types of withdrawals, and special rules apply to certain types of accounts.
- **Inheritance:** Either a flat tax or progressive tax rates are assessed on the inheritance above an exemption, with rates and exemptions depending on the relation of the recipient to the deceased and their disability status. Transfers to spouses/civil partners are exempt. Certain shares are required to pass to the deceased's children.

Germany

- **Wealth:** No wealth tax.
- **Capital gains:** Generally a flat tax is assessed on gains above an exemption, but taxpayers with lower labor income can opt to apply their labor income tax rate instead. Gains from certain classes of assets are exempt or subject to special rules.
- **Property:** A flat tax is assessed on property, with rates depending on the class of property and subject to a multiplier, which varies by locality.
- **Pensions:** No tax on capital gains made within the pension account. A portion of pension benefits, which depends on the type of account, is subject to the same progressive tax rates as labor income.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above an exemption, with tax rates and exemptions both depending on the relation of the recipient to the deceased. Pension entitlements are exempt.

Ireland

- **Wealth:** No wealth tax.
- **Capital gains:** A flat tax is assessed on gains above an exemption, with the rate depending on the type of asset. Certain classes of individuals, such as farmers and entrepreneurs, qualify for lower rates and additional exemptions.
- **Property:** Progressive tax rates are assessed on residential properties, with local authorities able to vary the rates to a certain extent. A flat tax is assessed on commercial properties, with rates varying by locality.
- **Pensions:** No tax on capital gains made within the pension account. Depending on the type of withdrawal, pension benefits are either subject to the same progressive tax rates as labor income or different progressive tax rates beyond an exemption. A surtax is assessed on high-value accounts.
- **Inheritance:** A flat tax is assessed on inheritances above an exemption. Exemptions are associated with the recipient and apply to the sum of all inheritances bequeathed to the recipient from certain classes of relatives.

Israel

- **Wealth:** No wealth tax.
- **Capital gains:** Generally a flat tax is assessed on real gains (i.e., the inflationary component of gains is exempt). High-income individuals are subject to a surtax.
- **Property:** Generally the tax increases in the area of the property, with amounts depending on property characteristics and varying by municipality. Tax relief may apply to certain taxpayers, such as new immigrants and low-income individuals, depending on the municipality.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income; certain taxpayers qualify for exemptions.
- **Inheritance:** No inheritance tax.

Italy

- **Wealth:** A flat tax is assessed on bank deposits and financial investments held abroad, with exemptions on bank deposits if the average annual account balance is below a certain threshold.

- **Capital gains:** Generally a flat tax is assessed on financial capital gains. For certain real estate capital gains, individuals can choose between separable or composite taxation, either applying a flat tax or their labor income tax rate.
- **Property:** Generally a flat tax is assessed on property, with rates depending on property characteristics and varying by municipality.
- **Pensions:** A flat tax is assessed on capital gains made within the pension account, with the rate depending on the type of asset. Pension benefits are also subject to flat taxes, with rates varying with the duration of the contribution period.
- **Inheritance:** A flat tax is assessed on inheritances, with higher rates for more distant relatives. Different amounts of the inheritance are exempt from taxation for certain close relatives.

Japan

- **Wealth:** No wealth tax.
- **Capital gains:** A flat tax is assessed on gains from certain classes of assets, such as securities and real estate, with the rate depending on the type of asset. Progressive tax rates, composite taxation, exemptions, and discounts apply to gains from different classes of assets.
- **Property:** A flat tax is assessed on property above an exemption, with a lower rate or reduction applicable to certain types of property.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to progressive tax rates, with the rates depending on the type of withdrawal.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above a general exemption and an exemption that depends on the relation of the recipient to the deceased and their disability status. A surtax applies to more distant relatives. Certain shares are required to pass to certain relatives.

Netherlands

- **Wealth:** A progressive, fictitious estimated return from net assets not intended for daily use is taxed at a flat rate depending on the amount above the exemption.
- **Capital gains:** Gains from a company in which an individual has a substantial stake are subject to a flat tax. Most other capital gains are not subject to taxation.
- **Property:** At the municipal level, a flat tax is assessed on property, with rates depending on property characteristics and varying by municipality. At the national level, progressive tax rates are assessed on the fictitious estimated rental values of primary residences, with

substantial deductions applicable to the portion of the tax exceeding the mortgage interest deduction.

- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income, though certain accounts with taxed contributions allow tax-free withdrawals.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above an exemption, with tax rates and exemptions depending on the relation of the recipient to the deceased and their disability status. Additional exemptions apply to certain classes of assets.

New Zealand

- **Wealth:** No wealth tax.
- **Capital gains:** Capital gains from financial assets are generally either subject to composite taxation or are exempt from taxation, depending on the type of gain. Special rules apply to certain classes of assets. Capital gains from real estate are generally subject to composite taxation. Depending on transaction characteristics, gains from the sale of commercial property may be subject to an additional tax, while gains from the sale of residential property may be exempt from taxation.
- **Property:** Generally a fixed fee plus a flat tax is assessed on property, with rates set at the municipal level. Low-income individuals qualify for rebates for owner-occupied residential property.
- **Pensions:** A flat tax is assessed on capital gains made within the pension account, with the rate depending on the type of account; for certain accounts, the rate depends on the taxpayer's labor income in prior years. Pension benefits are generally exempt from taxation.
- **Inheritance:** No inheritance tax.

Norway

- **Wealth:** A flat tax is assessed on wealth above an exemption, with the value of certain classes of assets, such as primary and secondary residences, discounted.
- **Capital gains:** A flat tax is assessed on gains from financial assets above the "risk-free" return (i.e., the counterfactual return on treasury bills of the same value). Gains from certain financial assets, such as dividends, are multiplied by a factor before the tax is assessed. A flat tax is assessed on real estate gains, with exemptions for certain types of property.
- **Property:** A flat tax is assessed on discounted property values, with rates varying by municipality and discounts varying by property type.

- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to a lower tax rate than labor income, and taxpayers with smaller benefits qualify for larger tax deductions.
- **Inheritance:** No inheritance tax.

Portugal

- **Wealth:** No wealth tax.
- **Capital gains:** Generally a flat tax is assessed on gains from financial assets, but for certain types of gains, such as interest, low-income individuals can opt to apply their labor income tax rate. For real estate capital gains, a discount is first applied to the gain and then the discounted gain is added to labor income and taxed progressively. Certain classes of real estate are exempt.
- **Property:** Progressive tax rates are assessed on property, with exemptions for certain taxpayers. Rates and exemptions vary based on property characteristics, and an additional exemption applies to low-income individuals.
- **Pensions:** No tax on capital gains made within the pension account, except for dividends, which are generally subject to a flat tax. For different types of withdrawals above an exemption, capital gains are either subject to a flat tax or the same progressive tax rates as labor income when withdrawn. Depending on how contributions were initially taxed and the type of withdrawal, the non-capital gains component of benefits is exempt from taxation, or subject to a flat tax or the same progressive tax rates as labor income on the amount above an exemption.
- **Inheritance:** A flat tax is assessed on the inheritance, with a higher rate for real estate transfers. Transfers to spouses/civil partners, ascendants, and descendants are exempt (except for real estate transfers, which are subject to a low flat tax).

Singapore

- **Wealth:** No wealth tax.
- **Capital gains:** Most capital gains are not subject to taxation. Depending on transaction characteristics, composite taxation may apply.
- **Property:** Progressive tax rates are assessed on the estimated rental value of the property, with rates varying by property type and occupancy status.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are generally subject to the same progressive tax rates as labor income; benefits from contributions made before a certain year are exempt from taxation.

- **Inheritance:** No inheritance tax.

South Korea

- **Wealth:** No wealth tax.
- **Capital gains:** Various flat and progressive tax rates are assessed on gains above an exemption; rates and exemptions depend on the type of asset. Gains from certain classes of assets are entirely exempt. Dividends and interest are subject to flat taxation below a certain limit and composite taxation above that limit.
- **Property:** Progressive tax rates are assessed on property, with rates varying by property type.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits beyond a progressive exemption (i.e, greater portions are exempt at smaller benefit levels) are generally subject to the same progressive tax rates as labor income; the exempt amount may also depend on the type of withdrawal and taxpayer characteristics.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above either a lump-sum or itemized deduction, which depends on the composition of the inheritance and relation of the recipient to the deceased. Transfers to spouses are exempt. The top tax rate increases for controlling shares in a company.

Spain

- **Wealth:** Progressive tax rates are assessed on net assets above an exemption, with an additional exemption for residences.
- **Capital gains:** Progressive tax rates are generally assessed on gains, with exemptions for elderly individuals under certain conditions and for certain real estate gains.
- **Property:** Generally a flat tax is assessed on property, with rates depending on the property type and varying by locality. Exemptions or discounts may apply depending on taxpayer and property characteristics, including taxpayer income.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are subject to the same progressive tax rates as labor income.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above an exemption, with tax rates and exemptions depending on the relation of the recipient to the deceased and their disability status. Certain classes of assets, such as family businesses and art collections, are eligible for additional exemptions.

Switzerland

- **Wealth:** A flat tax is assessed on the net value of certain classes of assets and liabilities, with tax rates and exemptions varying by canton.
- **Capital gains:** Progressive tax rates are assessed on gains from real estate, with rates varying by canton. Most capital gains from financial assets are not subject to taxation. Dividends and interest are subject to composite taxation.
- **Property:** Generally a flat tax is imposed on property, with rates varying by canton; a minimum amount per property may apply. For owner-occupied properties not rented out, an estimated rental value is subject to composite taxation.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits are subject to either the same progressive tax rates as labor income or lower progressive tax rates, depending on the type of withdrawal.
- **Inheritance:** In most cantons, progressive tax rates are assessed on the inheritance and depend on the relation of the recipient to the deceased. Transfers to spouses and children are exempt in most cantons.

Taiwan

- **Wealth:** No wealth tax.
- **Capital gains:** Most capital gains from financial assets are subject to composite taxation; taxpayers can opt for a flat tax to be assessed on dividends, and certain gains are exempt from taxation. A flat tax is assessed on gains from real estate, with the rate depending on the type of asset, and an exemption for primary residences.
- **Property:** Flat or progressive tax rates are assessed on land, depending on its intended use. A flat tax is generally assessed on buildings, with rates depending on their intended use. Certain classes of land and buildings are exempt or subject to reduced rates.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits beyond an exemption—which depends on the duration of the contribution period—are subject to the same progressive tax rates as labor income.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above an exemption, which depends on the relation of the recipient to the deceased, their disability status, and their age.

United Kingdom

- **Wealth:** No wealth tax.

- **Capital gains:** Either flat or progressive tax rates are assessed on gains, with rates depending on the taxpayer's labor income tax bracket; higher rates generally apply to taxpayers in higher labor income tax brackets. Exemptions for part or all of the gain apply to certain types of assets, such as dividends and primary residences.
- **Property:** Progressive tax rates are assessed on property, with rates varying by locality. Exemptions or discounts may apply to certain taxpayers depending on characteristics, such as age.
- **Pensions:** No tax on capital gains made within the pension account. Pension benefits beyond an exemption are subject to the same progressive tax rates as labor income. An additional flat tax may be imposed on accounts with a value exceeding a lifetime limit, with the tax rate depending on the type of withdrawal.
- **Inheritance:** A flat tax is assessed on the inheritance above an exemption, with larger exemptions for transfers to children. Transfers to spouses/civil partners, charities, and amateur sports clubs are exempt. The tax rate is reduced if a certain share is transferred to charity.

United States

- **Wealth:** No wealth tax.
- **Capital gains:** Gains from "short-term" assets (held for less than a year) are subject to composite taxation. Gains from "long-term" assets are subject to a flat tax, with higher rates for higher-income individuals. Dividends are also subject to either composite taxation or flat taxes that increase with labor income, depending on their source.
- **Property:** Generally a flat tax is assessed on property, with rates varying by state, county, and municipality.
- **Pensions:** No tax on capital gains made within the pension account. Depending on the type of account, benefits are generally either exempt from taxation or subject to the same progressive tax rates as labor income.
- **Inheritance:** Progressive tax rates are assessed on the inheritance above an exemption. Transfers to spouses are generally exempt.