A Welfare Comparison of Ad Valorem and Specific Taxes in Multi-Product Markets*

Adam Dearing†

The Ohio State University

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Abstract

We examine the efficiency ranking of ad valorem vs. specific taxes in multi-product markets using a general framework that admits flexible substitution patterns, various ownership structures and conduct, and marginal cost heterogeneity. First, we consider a single tax rate applied to all products. With symmetric costs, we provide new results regarding dominance of ad valorem taxes. With asymmetric costs, we provide a useful sufficient conditions for the specific tax to welfare dominate and discuss when this is more ore less likely under various forms of conduct.

We also provide welfare comparison for good-specific tax rates on substitutable goods. When firms have market power, we show that the ad valorem taxes dominate, even with arbitrary cost heterogeneity.

Our results suggest that both market power and the ability to target tax rates to specific goods support ad valorem taxes, while certain forms of cost heterogeneity work in favor of specific taxes.

Keywords: Ad valorem tax, specific tax, welfare, market power, multi-product oligopoly

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*All errors are our own.

†Department of Economics, 410 Arps Hall, 1945 N. High St., Columbus OH 43210, United States. Email: dearing.30@osu.edu.

1 Introduction

Ad valorem (percentage) and specific (unit) taxes generate revenue for governments around the world, and efficiency comparison of these forms of taxation has long been an area of interest for economists. In homogeneous goods markets, it is well-known that the ad valorem and specific taxes are welfare-equivalent under perfect competition. However, this result does not hold under imperfect competition. Suits and Musgrave (1953) provided the first formal proof that ad valorem taxation welfare-dominates specific taxation under monopoly. This result was extended to symmetric homogeneous product oligopoly markets by Delipalla and Keen (1992), who use a conjectural variations approach to show that the ad valorem tax welfare-dominates. Anderson, de Palma and Kreider (2001) show that the same result holds for asymmetric Cournot, where firms have heterogeneous costs.

Skeath and Trandel (1994) strengthen the argument in favor of ad valorem taxes in terms of Pareto dominance, a more strict criterion than welfare dominance. They show that the ad valorem tax Pareto dominates the ad valorem tax in a single-products monopoly. This result does not apply to imperfect competition in general; they also show that neither tax can be guaranteed to Pareto dominate the other under Cournot competition.

In differentiated products markets, the analysis becomes more complex and results in the literature are somewhat more mixed. Anderson, de Palma and Kreider (2001) provide some results in this context context. Under homogeneous, constant marginal costs, they show that the ad valorem tax welfare-dominates when firms have market power. This result does not hold up under heterogeneous costs, where they are able to provide a stylized example of Bertrand competition in which the specific tax welfare-dominates. But they do not provide a more general analysis under heterogeneous costs, neither for Bertrand nor for other forms of competition.

Since Anderson, de Palma and Kreider (2001), work on welfare comparisons of the tax types in differentiated-products models with heterogeneous costs has mainly focused on stylized models. Wang and Zhao (2009) consider a symmetric market where the utility function is quadrated in quantities, so (inverse) demand is linear. In this parameterized setting, they examine Bertrand and Cournot competition and provide conditions under which either type of tax can welfare-dominate, stating, "... if the substitution coefficient is sufficiently small, the cost variance is sufficiently and the ad valorem tax rate is sufficiently high then unit taxation can be welfare superior ...” (pp. 235). Wang and Wright (2017) take a different approach, assuming no substitution across products, focusing on perfect competition, and explicitly tying demand to marginal cost in a particular way. In their setting, they find that the ad valorem tax welfare-dominates because it induces larger effective taxes on the less elastic goods, meaning it is closer to the Ramsay optimal tax principal.

The primary focus of this paper is on providing a much more general analysis of tax efficiency in differentiated products markets. Our analysis proceeds in a general framework that allows for flexible substitution patterns amongst products, arbitrary ownership structures, and several different forms of competition, including perfect competition, monopoly, Cournot, and Bertrand competition. Our analysis is broken up into two settings: broad-based taxes that apply to all goods simultaneously, and good-specific taxes.

Under a single, broad-based tax, we make several new contributions to the literature. First, we provide the first results for Pareto comparisons of ad valorem and specific in differentiated products markets. Second, we provide sufficient conditions for specific taxes to welfare dominate when costs are heterogeneous. We show that market power makes the condition more stringent in a certain sense, indicating that the presence of imperfect competition will tend to favor ad valorem taxation, even when costs are heterogeneous. Our general results also provide context for those of Anderson, de Palma and Kreider (2001), Wang and Zhao (2009), and Wang and Wright (2017) in more limited models.

We also provide analysis in when there are multiple, good-specific tax rates. To our knowledge, we are the first in the literature to compare efficiency of good-level ad valorem and specific taxes when goods are substitutes. Here, we find that ad valorem taxation unambiguously welfare-dominates specific taxation when firms have market power, even with heterogeneous costs.

Broadly, our results suggest that even though it is possible specific taxation to welfare-dominate ad valorem taxation in differentiated products markets, this is less likely when firms have market power or when there is more ability to tailor taxes to individual goods.

In recent related work, Adachi and Fabinger (2018) perform a pass-through analysis of taxes in multi-product markets, substantially generalizing the analysis of Weyl and Fabinger (2013) for homogeneous products markets. They provide formulas relating pass-through to several economic quantities of interest, e.g. tax incidence and the marginal excess burden of public funds. But direct comparison of the welfare effects of ad valorem and specific taxation is limited to a single, broad-based tax in a symmetric market. We view their work as complementary to ours, since we provide much more robust results for the direct welfare comparison, but we do not consider incidence or general pass-through.

Also related to our paper, Agrawal and Hoyt (2019) consider tax pass-through in differentiated products markets, with particular focus on the difference between pass-through when a tax is broad-based vs. good-specific. Their work is also complementary to ours, since their focus is on pass-through and they do not perform welfare comparisons between ad valorem and specific taxes.

The remainder of the paper proceeds as follows. Section 2 describes our model of a multi-product market. Section 3 considers a single tax rate that applies to all goods. Section 4 looks at good-specific tax rates. Section 5 concludes. Proofs omitted from the main text appear in the appendix.

2 The Multi-Product Market

We consider a market with multiple substitutable products, indexed $j = 1, \ldots, J$. We assume that utility is quasi-linear in money and define the indirect utility function as $V(\boldsymbol{p}) = V(p_1, p_2, \ldots, p_J)$. The marginal utility of a dollar normalized to one (without loss of generality) so the units of $V(\cdot)$ are dollars. The assumption of quasi-linearity implies there are no income effects and is standard in the literature on relative efficiency of ad valorem and specific taxes. Single-product analysis usually uses consumer's surplus to measure consumer welfare, so this is a natural extension of that framework. With no income effects assumed, indirect utility has some useful properties.

Remark 1. $V(\cdot)$ has the following properties:

1. $V(\boldsymbol{p})$ is decreasing in p_j, $j = 1, \ldots J$.
2. $\partial V(\boldsymbol{p})/\partial p_j = -q_j(\boldsymbol{p})$ (Roy's Identity).
3. $V(\boldsymbol{p})$ is convex.
4. $d\boldsymbol{q}(\boldsymbol{p})/d\boldsymbol{p}$ is symmetric, negative semi-definite.

These properties of $V(\cdot)$ are the main tools used in the analysis in this paper. While they are standard results from consumer theory, they have so far not been utilized in assessing the relative efficiency of different types of taxes. By making use of them, we will be able to provide alternative proofs of established results and to provide several new results and insights into the tax efficiency.

On the supply side, we assume that each good j has constant marginal cost, c_j. This assumption is reasonable for many markets studied in empirical economics, and it is often used in the previous literature on tax efficiency in multi-product settings. We allow for multi-product firms and assume without loss of generality that products are ordered by firm. Indexing firms $f = 1, \ldots, F$ ($F \leq J$), we can then define the "ownership matrix" as

$$T = \begin{bmatrix} T_1 & 0 & 0 & 0 \\ 0 & T_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & T_F \end{bmatrix},$$

where T_f is a $J_f \times J_f$ matrix of ones and J_f is the number of products owned by firm f. This leads to convenient notation for the equilibrium conditions under various forms of competition.

Proposition 1. *Without taxes, the equilibrium condition is*

$$\boldsymbol{p} + \Omega \boldsymbol{q} = \boldsymbol{c},$$

where Ω is determined by the mode of competition:

- *Perfect competition:* $\Omega = 0$.
- *Monopoly:* $\Omega = \left(\frac{d\boldsymbol{q}(\boldsymbol{p})}{d\boldsymbol{p}}\right)^{-1}$
- *Bertrand competition:* $\Omega = \left(T * \frac{d\boldsymbol{q}(\boldsymbol{p})}{d\boldsymbol{p}}\right)^{-1}$
- *Cournot competition:* $\Omega = T * \left(\frac{d\boldsymbol{q}(\boldsymbol{p})}{d\boldsymbol{p}}\right)^{-1}$

In the latter three cases, Ω is negative definite. Furthermore, all elements of $\Omega \boldsymbol{q}$ are negative when $-\boldsymbol{q}(\boldsymbol{p})$ is inverse-isotone, so that markups are positive.

The equilibrium condition in Proposition 1 follows from firms' profit-maximizing first-order conditions under each type of conduct. When taxes are introduced, we will simply alter these conditions accordingly. The result that markups are positive when $-\boldsymbol{q}(\boldsymbol{p})$ is inverse-isotone captures the notion that markups are positive for substitutable products, but the weaker condition of inverse-isotonicity also allows for complementarity.

Throughout the analysis we will also assume that industry/market profits are concave in prices in the relevant ranges of prices, although we do not require profits to be concave for *all* possible prices. Our much weaker assumption is unlikely to be very restrictive in practice, and we can verify it at at least two outcomes of interest: the monopolist's profit-maximizing price/quantity and the competitive equilibrium. The monopolist's case is trivial – the monopolist maximizes industry profits; and it is easy to show that the Hessian of industry profits when $\boldsymbol{p} = \boldsymbol{c}$ is equal to $2d\boldsymbol{q}/d\boldsymbol{p}$, which is negative definite. In many markets, concavity over the entire relevant range of prices/quantities may be reasonable.

We also makes some restrictions on the substitution patterns of goods. We assume that goods are gross substitutes and that the substitution matrix is diagonally dominant, so that substitution effects are not "too" strong. This assumption and some of its implications are summarized below.

Assumption 1. *Products are gross substitutes and* $d\boldsymbol{q}/d\boldsymbol{p}$ *is diagonally dominant.*

Remark 2. Assumption 1 implies that the diagonal entries of $d\boldsymbol{q}/d\boldsymbol{p}$ are negative and its off-diagonals are non-negative. Additionally, Ω^{-1} has all non-positive entries and strictly negative diagonals under any of the definitions of Ω in Proposition 1.

Remark 3. An immediate consequence of Remark 2 is that all forms of imperfect competition in Proposition 1 result in strictly positive markups for all products.

Under Assumption 1, demand is inverse-isotone (Berry and Haile (2014)). This assumption is satisfied in commonly-used demand models, including the random coefficients logit model of Berry, Levinsohn and Pakes (1995), which has become a workhorse demand model for empirical work in multi-product markets. We will assume that Assumption 1 holds for the remainder of the analysis in the paper.

3 Single Tax Rate

In this section, we consider a single tax rate – either ad valorem or specific – that applies to all goods in the market. Such taxes are common, with one of the most salient examples being state-level sales taxes imposed in the United States. Other examples include ethanol taxes in alcoholic beverage markets and airline flight taxes (both ad valorem and specific).

We approach welfare comparisons in the framework of this question: if the benevolent government has a revenue target that it knows it could hit with either an ad valorem or a specific tax (or both), which one should it choose? By benevolent, we mean that the government maximizes total welfare – or perhaps just consumer welfare – subject to a revenue target constraint. This approach dates back to Ramsey (1927), who considered optimal taxation of perfectly competitive markets with income effects.

Throughout this section, we will denote the ad valorem tax v and specific tax τ. We denote the revenue collected from each tax as R_v and R_τ, respectively. We will measure welfare using total surplus: the sum of indirect utility, revenue, and industry/market profits.

3.1 Homogeneous Costs

We begin with the case of homogeneous costs: $c = c_1 = c_2 = \cdots = c_J$, so that $\boldsymbol{c} = \mathbf{1}c$. Here, we will use our framework to replicate prior results from the literature. In single-product monopoly, Suits and Musgrave (1953) were the first to prove that an ad valorem tax welfare dominates a specific tax. They show when the two taxes are revenue-equivalent, the ad valorem tax results in a higher quantity than the specific tax, leading to higher consumer and total surplus. Delipalla and Keen (1992) extend this result to homogenous products Cournot competition, Anderson, de Palma and Kreider (2001) extend this result to multi-product firms under various types of competition.

They key here is that homogeneous marginal costs induce sets of quantities (and prices) induced by the two taxes are identical – for every specific tax, there is a corresponding ad valorem tax that induces the same quantities of all goods. This comes from the equilibrium first-order conditions, which are given in the next proposition.

Proposition 2. *Taxes under specific tax, τ, and ad valorem tax, v, the equilibrium first-order conditions are, respectively,*

$$\boldsymbol{p} + \Omega \boldsymbol{q} - \mathbf{1}\tau = \mathbf{1}c$$

$$(\boldsymbol{p} + \Omega \boldsymbol{q})(1 - v) = \mathbf{1}c,$$

where Ω *is as in Proposition 1.*

Proposition 2 shows how taxes affect the equilibrium conditions. These first-order conditions imply that the two taxes will yield the same prices and quantities when $v = \tau/(c + \tau)$, because these lead to the same "effective marginal cost for firms": $\tilde{c} = c - \tau = c/(1 - v)$. Thus, the sets of quantities and prices induced by either tax are the same. From here, we can obtain a relationship between revenues of taxes that induce the same quantities.

Proposition 3. *Suppose* $v = \tau/(c + \tau)$*, so that* v *and* τ *induce the same equilibrium quantities and prices. Then,*

$$R_v + \boldsymbol{q}'\Omega \boldsymbol{q} v = R_\tau.$$

Proposition 3 establishes that ad valorem taxes dominate specific taxes under monopoly, Bertrand competition, and Cournot competition. Since Ω is negative definite in all three cases, the ad valorem tax collects more revenue than the welfare-equivalent specific tax. The proposition also establishes that the two taxes are welfare-equivalent in competitive equilibrium, with $\Omega = 0$.

So far, we have described efficiency in terms of the relationship between revenue and total surplus, but it is also interesting to consider the more stringent criterion of Pareto dominance. In homogenous products markets, Skeath and Trandel (1994) show that an ad valorem tax Pareto dominates a specific tax for a monopoly. They also show that Pareto dominance never holds when there are enough firms in the market for homogenous-products Cournot. Whether or not similar results hold in differentiated products settings is an open question, which we now address here. We find results similar to theirs: the ad valorem tax Pareto dominates the specific tax under monopoly, but this result is not guaranteed under imperfect competition.

Proposition 4. *Suppose that prices are increasing in the amount of either tax. Then, the ad valorem tax Pareto dominates the specific tax under monopoly when the revenue requirement is not too large.*

Proof. First, we note two facts: i) when there are no taxes, the monopolist's tax maximizes (pre-tax) industry profits; and ii) revenue is increasing in both tax rates when they are equal to zero. This means that when revenue requirements are not too large, two things happen: i) pre-tax profits are decreasing in both tax rates; and ii) after-tax profits are increasing. We proceed by showing that for a given specific tax, τ, there is an ad valorem tax that Pareto dominates it. We assume that R_τ is exactly equal to the revenue requirement. Proposition 3 shows that setting $v = \tau/(c+\tau)$ yields $R_v > R_\tau$. If the government decreases the ad valorem tax to $\tilde{v}'$ such that $R_{\tilde{v}} = R_\tau$, then utility increases because prices decrease. Pre-tax profit also increases relative to τ, so after-tax profit is also larger than under τ. Thus, $\tilde{v}$ is a Pareto improvement over τ. □

Proposition 5. *Suppose firms engage in Cournot competition and that equilibrium quantities are decreasing in the amount of both taxes. Then, there is no ad valorem tax that Pareto dominates τ when τ is sufficiently small.*

Proof. First, inverse-isotonicity of $-\boldsymbol{q}(\boldsymbol{p})$ ensures that all elements of $(d\boldsymbol{q}/d\boldsymbol{p})^{-1}$ are negative. With no taxes, Proposition 1 shows that the equilibrium condition for Cournot competition is $\boldsymbol{p} + \left(T * (d\boldsymbol{q}/d\boldsymbol{p})^{-1}\right)\boldsymbol{q} - \mathbf{1}c = 0$. The derivative of industry profits with respect to quantities is then $\boldsymbol{p} + (d\boldsymbol{q}/d\boldsymbol{p})^{-1}\boldsymbol{q} - \mathbf{1}c < 0$ because Hadamard multiplication by T. So, (pre-tax) industry profits are decreasing in all quantities near the without-tax Cournot equilibrium. For small enough τ, pre-tax industry profits will still be decreasing in quantities and government revenue will be increasing in tax rates. When $v = \tau/(c+\tau)$, prices and quantities are the same as at τ, so consumer welfare is the same. But $R_v > R_\tau$ by Proposition 3. So, industry level (after-tax) profits are lower than at $\tau$and some firm must be worse-off under v. For any $\tilde{v} > v$, quantities are lower and therefore prices are higher (because $-\boldsymbol{q}(\boldsymbol{p})$ is inverse-isotone), so consumer welfare is lower than at τ. For any $\tilde{v} < v$, pre-tax industry profits are lower than at τ (equivalently v). So, either $R_{\tilde{v}} \geq R_\tau$ and some firm must have lower after-tax profits, or $R_{\tilde{v}} < R_\tau$ and the government is worse-off. So, no ad valorem tax rate Pareto dominates τ. □

Propositions 4 and 5 provide multi-product generalizations of results in Skeath and Trandel (1994, Theorem 1 and Proposition 2, respectively) and are the first such results for differentiated products in the literature. The key element is whether raising the tax would result in an increase or decrease in pre-tax profits. In the former case, Pareto dominance of ad valorem taxes is unlikely, since there is a tradeoff between making firms worse-off and making consumers better-off when switching the type of tax used. Since the monopolist facing no taxes implements the pre-tax profit-maximizing price and quantity, it is likely that raising the tax rate will generally decrease pre-tax profits, since the prices/quantities move further from the optimum. So, even though the conditions for Proposition 4's result can only be guaranteed to hold locally, it is reasonable to think that they will hold for larger tax rates and revenue targets as well. Conversely, Cournot competition can be expected to produce quantities (prices) that are above (below) those that optimize pre-tax profits. Because making consumers better off requires lowering the tax, this will likely result in further reductions in pre-tax profits, preventing a Pareto improvement.

The homogeneous product setting of Skeath and Trandel (1994) does not admit Bertrand competition that differs from perfect competition, so they did not consider that case. However, our differentiated products setting allows us to extend the analysis to this case. At the very least, we can apply the same intuition as we did to the Cournot result, leading us to conjecture that a Pareto improvement will be difficult to obtain under Bertrand competition. Indeed, we are able to obtain a result similar to Proposition 5, although it turns out that Bertrand competition requires some more restrictions on the market to ensure that Pareto dominance cannot be guaranteed. Proving the result for Bertrand is much more technical, and we relegate the proof to the appendix.

Proposition 6. *Suppose that firms engage in Bertrand competition and that equilibrium prices are increasing in the amount of both taxes. Then, there is no ad valorem tax that Pareto dominates τ when τ is sufficiently small.*

Proposition 6 represents the first result of its kind for the Bertrand case. While the assumptions are more restrictive than those for the analogous result for Cournot competition, they are often satisfied in models used in empirical applications of differentiated products. For example, condition (i) is met in many discrete choice models, including the random coefficients logit model (Berry, Levinsohn and Pakes (1995)). Condition (ii) is generally to be expected and is often observed empirically.

3.2 Heterogeneous Costs

While the homogeneous cost assumption may be fairly reasonable for some heterogeneous products markets (e.g., soft drinks), it is broadly more natural to expect that heterogeneous products will have heterogeneous marginal costs. In this case, the results from the homogeneous costs analysis do not apply because the ad valorem and specific taxes no longer induce identical sets of quantities. Anderson, de Palma and Kreider (2001) examined this case, although their analysis is somewhat limited. For the homogeneous products Cournot model, they show that the ad valorem tax always welfare-dominates the specific tax. However, they show that this does not necessarily hold for differentiated-products Bertrand by providing a stylized example where the specific tax welfare-dominates the ad valorem tax.

Based on their theoretical results and simulation evidence, Anderson, de Palma and Kreider (2001) state that their analysis "(weakly) suggests that it is primarily the mode of competition that is responsible for the result" (pp. 243), but they do not provide a further general analysis for various forms of competition in differentiated products markets with heterogeneous costs. In this section, we provide such a general analysis that sheds further light on their results for Bertrand and Cournot, while providing additional results for multi-product monopoly and perfect competition. To do so, we first examine the case of perfect competition (zero markups) and establishing sufficient conditions for the specific tax to welfare-dominate the ad valorem tax. We then provide further sufficient conditions for the cases with market power and compare them.

We begin with perfect competition. This allows us to separate the roles of heterogeneous costs and market power in welfare results. First, we note that the conditions describing the equilibrium are nearly identical to Proposition 2, but with a more general marginal cost vector:

$$\boldsymbol{p} + \Omega \boldsymbol{q} - \boldsymbol{1}\tau = \boldsymbol{c}$$

$$(\boldsymbol{p} + \Omega \boldsymbol{q})(1 - v) = \boldsymbol{c},$$

with $\Omega = 0$ for perfect competition. Here, we can immediately see that the "equal effective costs" argument no longer holds. Effective costs are $c_j - \tau$ and $c_j/(1 - v)$, but if we have $c_j - \tau = c_j/(1 - v)$ for some j, then we will have $c_{j'} - \tau \neq c_{j'}(1 - v)$ whenever $c_{j'} \neq c_j$. With different effective costs, there will be different quantities induced by the two different types of taxes, and we will need to use different tools in the analysis.

Under perfect competition, total welfare is equal to indirect utility plus government revenue, since firms earn zero after-tax profits. To make welfare comparisons, we consider two taxes that produce the same level of indirect utility for consumers and compare the respective revenues they generate. We can also consider two taxes that generate the same revenue and compare their indirect utilities. Either way, obtain sufficient conditions for either tax to welfare-dominate the other. For notational convenience, we define the indirect utility and revenue of the ad valorem (specific) tax as V_v and R_v (V_τ and R_τ), respectively. We also denote the induced prices and quantities in a similar manner: $\boldsymbol{p}_v$ and $\boldsymbol{q}_v$ ($\boldsymbol{p}_\tau$ and $\boldsymbol{q}_\tau$).

Proposition 7. *Suppose that $V_v = V_\tau$ and $\boldsymbol{p}_v \neq \boldsymbol{p}_\tau$ (and therefore $\boldsymbol{q}_v \neq \boldsymbol{q}_\tau$). Then $\sum_{j=1}^J q_{jv} \leq \sum_{j=1}^J q_{j\tau}$ is sufficient for $R_v < R_\tau$, and $\sum_{j=1}^J p_{jv} q_{jv} \geq \sum_{j=1}^J p_{jv} q_{j\tau}$ is sufficient for $R_v > R_\tau$.*

Proof. First, it is straightforward to show that under perfect competition, $\boldsymbol{p}_\tau - \boldsymbol{p}_v = \mathbf{1}\tau - \boldsymbol{p}_v v$. To prove that $\sum_{j=1}^J q_{jv} \leq \sum_{j=1}^J q_{j\tau}$ is sufficient for $R_v < R_\tau$, notice that convexity of the indirect utility function and Roy's identity imply that $V_\tau > V_v - \boldsymbol{q}_v'(\boldsymbol{p}_\tau - \boldsymbol{p}_v)$. Using $V_v = V_\tau$ and $\boldsymbol{p}_\tau - \boldsymbol{p}_v = \mathbf{1}\tau - \boldsymbol{p}_v v$, we then obtain $0 > -\boldsymbol{q}_v'(\mathbf{1}\tau - \boldsymbol{p}_v v)$. If $\sum_{j=1}^J q_{jv} \leq \sum_{j=1}^J q_{j\tau}$, so that $\boldsymbol{q}_v'\mathbf{1} \leq \boldsymbol{q}_\tau'\mathbf{1}$, then we have

$$\begin{aligned} R_v &= \boldsymbol{q}_v'\boldsymbol{p}_v v \\ &< \boldsymbol{q}_v'\mathbf{1}\tau \\ &\leq \boldsymbol{q}_\tau'\mathbf{1}\tau \\ &= R_\tau. \end{aligned}$$

Proving that $\sum_{j=1}^J p_{jv} q_{jv} \geq \sum_{j=1}^J p_{jv} q_{j\tau}$ is sufficient for $R_v > R_\tau$ is similar, but is based on convexity if indirect utility and Roy's identity implying that $V_v > V_\tau - \boldsymbol{q}_\tau'(\boldsymbol{p}_v - \boldsymbol{p}_\tau)$. With some manipulation, the condition appears. □

Proposition 8. *Suppose that $R_v = R_\tau$ and $\boldsymbol{p}_v \neq \boldsymbol{p}_\tau$ (and therefore $\boldsymbol{q}_v \neq \boldsymbol{q}_\tau$). Then $\sum_{j=1}^J q_{jv} \leq \sum_{j=1}^J q_{j\tau}$ is sufficient for $V_v < V_\tau$, and $\sum_{j=1}^J p_{jv} q_{jv} \geq \sum_{j=1}^J p_{jv} q_{j\tau}$ is sufficient for $V_v > V_\tau$.*

Propositions 7 and 8 establish sufficient conditions for one tax to welfare-dominate the other under perfect competition. Notably, the sufficient conditions are the same, whether we start with equal utility or with equal revenue for both taxes. The sufficient condition for the specific tax to dominate is that it induces a higher aggregate quantity than the ad valorem tax. This condition is novel and results from convexity of the indirect utility function. The sufficient condition for the ad valorem tax to dominate is that the quantities induced by the ad sufficient tax can be afforded at the ad valorem tax's prices. That condition also results from convexity of indirect utility, although it does not improve upon results from standard revealed preference arguments and is actually uninformative when $V_v = V_\tau$.[1] However, we still explicitly state the condition because its counterparts for imperfectly competitive markets will be be more informative.

Focusing on the sufficient condition for the specific tax to welfare-dominate is informative and helps explain some previous results for certain models in the literature. Wang and Wright (2017) study a particular model with multiple non-substitutable goods all subject to the same tax rate. In their model, higher-cost goods are more desirable, leading to lower demand elasticity. In their model, they find that the ad valorem tax is more efficient because it results in higher tax rates for lower-elasticity goods. We formally state this in the next proposition.

Proposition 9. *Suppose that $V_v = V_\tau$ and (without loss of generality) $c_1 \leq c_2 < \cdots \leq c_J$, with strict inequality between at least two marginal costs. Then, inducing $\boldsymbol{q}_v$ (and $\boldsymbol{p}_v$) with a vector of specific taxes, $\tilde{\boldsymbol{\tau}}$, requires that $\tilde{\tau}_1 < \tau$ and $\tilde{\tau}_J > \tau$. That is, the effective tax rate on the lowest-cost is lower under ad valorem taxation than specific taxation, while the effective tax rate on the highest-cost good is higher.*

Proof. We have $\boldsymbol{p}_v = \boldsymbol{c}/(1-v) = \boldsymbol{c} + \boldsymbol{c}v/(1-v)$ and $\boldsymbol{p}_{\tilde{\tau}} = \boldsymbol{c} + \tilde{\boldsymbol{\tau}}$. So, $\boldsymbol{p}_v = \boldsymbol{p}_{\tilde{\tau}}$ requires $\tilde{\tau}_j = c_j v/(1-v)$ for all $j = 1, \ldots, J$. The right-hand side of this last expression is increasing in c_j. If $\tilde{\tau}_1 \geq \tau$, then $\tilde{\tau}_j \geq \tau$ for each j, with strict inequality for some j. Therefore, $p_{vj} = p_{\tilde{\tau}j} \geq p_{\tau j}$, with strict inequality for some j, implying $V_v < V_\tau$ which is a contradiction. A similar argument shows that $\tilde{\tau}_J \leq \tau$ implies $V_v > V_\tau$, which is also a contradiction. □

Proposition 9 indicates that – compared to specific taxation – we can expect higher-cost goods to receive higher effective tax rates under ad valorem taxation, while lower-cost goods receive lower effective tax rates.

[1] By a standard revealed preference argument, $\sum_{j=1}^{J} p_{jv} q_{jv} \leq \sum_{j=1}^{J} p_{jv} q_{j\tau}$ is sufficient for $V_v > V_\tau$, assuming the utility-maximizing bundle is unique. So, the condition cannot apply when $V_v = V_\tau$.

[Results for imperfect competition coming soon . . . .]

4 Good-Specific Tax Rates

We now turn attention to a case where there are good-specific tax rates, with each good subject to its own specific or ad valorem rate. Here, the results will be the same with either homogeneous or heterogeneous costs, so will will consider (potentially) heterogeneous costs. When considering products that are not substitutes, ad valorem taxes will welfare-dominate because each good can be treated as having its own independent market. Ad valorem taxes also dominate with single-product firms even when products are substitutes, since the equilibrium conditions become $(p_j + \Omega_{jj}q_j)(1 - v_j) - \tau_j = c_j$ for each j, so an argument based on good-specific effective tax rates will yield welfare-dominance of the ad valorem tax. However, the situation becomes less clear when there are multi-good firms and substitution between goods. To our knowledge, ours is the first efficiency comparison between ad valorem and specific taxes in the literature for this case.

The difficulty with the case relative to the others arises because we cannot simply use an effective marginal cost-style argument like we can in the other cases. This is because the equilibrium first-order conditions do not have as convenient of a form as they do in the other cases. The issue comes with how ad valorem taxes affect the conditions, and we cannot do something as simple as divide the equation for each good by that good's tax rate.

Proposition 10. *With good-specific tax rates, the equilibrium conditions under specific and ad valorem taxation are, respectively,*

$$\boldsymbol{p} + \Omega\boldsymbol{q} - \boldsymbol{\tau} = \boldsymbol{c}$$

$$diag(\boldsymbol{p})(\boldsymbol{1} - \boldsymbol{v}) + \Omega\, diag(\boldsymbol{q})(\boldsymbol{1} - \boldsymbol{v}) = \boldsymbol{c},$$

where $diag(\boldsymbol{x})$ returns a diagonal matrix with diagonal entries corresponding to the elements of $\boldsymbol{x}$ and Ω is defined in the same way as Proposition 1.

Proof. Follows from firms' first-order conditions and using the fact that $\boldsymbol{p} * (\boldsymbol{1} - \boldsymbol{v}) = \text{diag}(\boldsymbol{p})(\boldsymbol{1} - \boldsymbol{v})$ and $\Omega(\boldsymbol{q} * (\boldsymbol{1} - \boldsymbol{v})) = \Omega\text{diag}(\boldsymbol{q})(\boldsymbol{1} - \boldsymbol{v})$. □

Proposition 10 gives the equilibrium conditions and illustrates the obstacle to attempting an effective cost-based argument, with non-zero off-diagonal elements in Ω being the source of the issue. However, our analysis in Section 3.1 for single tax rates and homogeneous costs revealed that the salient effective cost argument was that it implied the ad valorem tax could induce the same set of quantities and prices as the specific tax. This is also true for the good-specific tax rates, regardless of the cost structure, although the proof is a bit more involved.

Proposition 11. *For any combination of non-negative specific tax rates, there exists a combination of non-negative ad valorem tax rates that induces the same quantities and prices.*

Proof. We will show constructively that there is some $\boldsymbol{v} \geq 0$ that satisfies the equilibrium conditions at the quantities and prices induced by $\boldsymbol{\tau} \geq 0$. Close inspection of the equilibrium conditions in Proposition 10 shows that it is sufficient to find some $\boldsymbol{v}$ such that

$$(\text{diag}(\boldsymbol{p}) - \Omega\text{diag}(\boldsymbol{q}))\,\boldsymbol{v} = \boldsymbol{\tau}.$$

To prove that such a $\boldsymbol{v} \geq 0$ exists, it is sufficient to show that $\text{diag}(\boldsymbol{p}) - \Omega\text{diag}(\boldsymbol{q})$ is invertible and that its inverse has all non-negative elements, which we do by showing it is an M-matrix. First, note that all elements of $\Omega\text{diag}(\boldsymbol{q})$ are non-positive since all elements of Ω are non-positive. So, all off-diagonal elements of $\text{diag}(\boldsymbol{p}) - \Omega\text{diag}(\boldsymbol{q})$ are non-positive. Second, the equilibrium condition for the specific tax rate can be expressed as $(\text{diag}(\boldsymbol{p}) - \Omega\text{diag}(\boldsymbol{q}))\,\mathbf{1} = \boldsymbol{c} + \boldsymbol{\tau}$. Since the right-hand side has positive elements, the left hand side must have positive elements as well. The form of the left-hand side and with the fact that $\text{diag}(\boldsymbol{p}) - \Omega\text{diag}(\boldsymbol{q})$ has non-positive off-diagonal entries then immediately imply that $\text{diag}(\boldsymbol{p}) - \Omega\text{diag}(\boldsymbol{q})$ has positive diagonal entries and is strictly diagonally dominant. Therefore, it is an M-matrix and $(\text{diag}(\boldsymbol{p}) - \Omega\text{diag}(\boldsymbol{q}))^{-1}$ has all non-negative entries. □

With this result in place, we can now show that ad valorem taxes welfare-dominate specific taxes when they are good-specific.

Proposition 12. *For any specific tax rates, $\boldsymbol{\tau}$, there exist ad valorem tax rates, $\boldsymbol{v}$ that induce the same equilibrium quantities and prices, and*

$$R_{\boldsymbol{v}} + \boldsymbol{q}'\Omega\, diag(\boldsymbol{q})\boldsymbol{v} = R_{\boldsymbol{\tau}}.$$

Proposition 12 shows that $R_{\boldsymbol{v}} > R_{\boldsymbol{\tau}}$ when there is market power: $\Omega \leq 0$ has strictly negative diagonals and $\boldsymbol{v}$ is non-negative by Proposition 11, with at least one strictly positive element. To our knowledge, we are the first to obtain this result for even a multi-product monopolist, and it applies to broad ownership structures and modes of competition.

5 Conclusion

To be written . . . .

References

Adachi, T., and M. Fabinger. 2018. "Multi-Dimensional Pass-Through and Welfare Measures under Imperfect Competition." Working Paper. arXiv 1702.04967v3.

Agrawal, D.R., and W.H. Hoyt. 2019. "Pass-Through in a Multiproduct World." Working Paper.

Anderson, S.P., A. de Palma, and B. Kreider. 2001. "The Efficiency of Indirect Taxes Under Imperfect Competition." *Journal of Public Economics*, 81(2): 231–251.

Berry, S., J. Levinsohn, and A. Pakes. 1995. "Automobile Prices in Market Equilibrium." *Econometrica*, 63(4): 841–890.

Berry, S.T., and P.A. Haile. 2014. "Identification in Differentiated Products Markets Using Market Level Data." *Econometrica*, 82(5): 1749–1797.

Delipalla, S., and M. Keen. 1992. "The Comparison Between Ad Valorem and Specific Taxation Under Imperfect Competition." *Journal of Public Economics*, 49(3): 351–367.

Ramsey, F.P. 1927. "A Contribution to the Theory of Taxation." *The Economic Journal*, 37(145): 47–61.

Skeath, S.E., and G.A. Trandel. 1994. "A Pareto Comparison of Ad Valorem and Unit Taxes in Noncompetitive Environments." *Journal of Public Economics*, 53(1): 53–71.

Suits, D.B., and R.A. Musgrave. 1953. “Ad Valorem and Unit Taxes Compared.” *The Quarterly Journal of Economics*, 67(4): 598–604.

Wang, H., and J. Zhao. 2009. “On the Efficiency of Indirect Taxes in Differentiated Oligopolies with Asymmetric Costs.” *Journal of Economics*, 96(3): 223–239.

Wang, Z., and J. Wright. 2017. “Ad Valorem Platform Fees, Indirect Taxes, and Efficient Price Discrimination.” *RAND Journal of Economics*, 48(2): 467–484.

Weyl, E.G., and M. Fabinger. 2013. “Pass-Through as an Economic Tool: Principles of Incidence under Imperfect Competition.” *Journal of Political Economy*, 121(3): 528–583.

A Proofs Omitted from the Main Text

Coming soon . . . .