No Regret Fiscal Reforms:

Pierre-Edouard Collignon*

CREST - Ecole polytechnique, France;

July 15, 2021

Abstract

How should taxes react to shocks *ex post* while preserving incentives to work and save *ex ante*? The standard solution involves a commitment to a contingent policy which is unrealistic. As an alternative, this paper introduces "No Regret fiscal reforms" which are policy changes such that households do not regret their past decisions. Hence flexibility is provided and incentives to work and save are preserved. Such reforms can be achieved by changing taxes on both capital and labour such that wealth effect exactly compensates substitution effects. Optimal No Regret fiscal reforms are compared to optimal contingent policies in a representative agent framework. First, when shocks and their distributions are common knowledge, optimal No Regret reforms only lead to small welfare losses. Second, I consider Near-Rational Expectations i.e the government recognizes that agents' beliefs about the distribution of shocks may be different from its own and wants to implement a policy robust to this unknown difference. Then welfare differences are reduced and even disappear when the government wants a policy robust to large enough differences between the agents' and the government's distributions. Finally, introducing wealth and skill heterogeneity, I establish the existence of No Regret reforms.

Keywords: Ramsey model, stochastic public spending, fiscal rule, discretion.
JEL Classification: E61, E62, H21, H63

*pierre-edouard.collignon@polytechnique.edu - I am grateful to Pierre Boyer, Julien Combe, Alessandro Riboni, and Jean-Baptiste Michau, my PhD supervisor, for all our helpful discussions. I thank the audience at CREST seminar and at the GT Fiscalité.

1 Introduction

How should labour and capital taxes react to public spending shocks ? There exist two main approaches in the literature. The first is to opt for full discretion which allows governments to adapt to foreseen and unforeseen shocks alike. Unfortunately, with full discretion successive governments heavily tax capital and government bonds which is equivalent to a default. Optimal time-consistent plans are highly inefficient because the policy they involve very high capital taxes and low capital levels which are rarely observed empirically. The second main approach is to initially commit to a contingent policy. Optimal contingent plans involve low expected capital taxes and high capital accumulation. However, optimal contingent policies are complex to write and are not used in practice. This paper's approach is halfway between these two extremes, each government may *partially bind* future governments and each government has *some discretion* left. This approach is implemented as follows: the initial government defines a non-contingent policy, successive governments may change this policy using *No Regret fiscal reforms* defined as policy changes such that households do not regret any of their past decisions.

Take a two period setting as an example. At t=0, the government announces a non-contingent policy. At t=1, because of a war for example, the usefulness of public spending relative to private consumption becomes very large. The benevolent government would like to increase taxes so as to increase public spending. If the capital tax is increased, then agents regret having saved as much at t=0. If the labour tax is raised, then agents becomes poorer and regret having as little at t=0. There are many ways to increase (or decrease) both taxes so that agents don't regret their saving decisions at t=0. This is how No Regret reforms provide flexibility to the government to cope with shocks at t=1.

This paper uses an infinitely-lived representative household model with capital. The benevolent government taxes labour,capital and bonds linearly. Preferences for public spending are stochastic. There are no other shocks. I study how governments optimally use No Regret reforms to adapt to these shocks. I show that the labour tax is smoother than the government expenditures and that the capital and bond taxes are such that saving decisions are hardly distorted. I make comparisons with the optimal contingent policy benchmark. The main difference lies in the fact that contingent policies use contingent bonds which offer insurance against high spending shocks e.g. a contingent debt with low payoff if a war breaks out. Such insurance is not available to a government using No Regret reforms because households would regret having bought these contingent bonds when payoffs are low. This lack of insurance pushes government to self-insure through asset accumulation. As a result, the economy converges to the no-tax first best.[1] This contrasts with the optimal contingent policies

[1]In the same spirit of limiting insurance provided by government bonds, Aiyagari, Marcet, Sargent,

which use contingent bonds to get insurance. Then there is no asset accumulation by the government and tax distortion remains constant over time. This insurance fully explains the higher welfare achieved by contingent policies. However, No Regret reforms still provide much flexibility to governments to adapt to shocks *ex post*. To cope with higher (lower) spending needs, governments may increase (decrease) taxes, run a deficit (surplus) and *indirectly* make the capital stock decrease (increase). These three mechanisms provide important welfare gains when comparing to non-contingent policies ($\approx 1.20\%$ in consumption equivalent). The welfare loss compared to the optimal contingent policy is very small ($< 0.01\%$ in consumption equivalent).

Second, I consider Near-Rational Expectations *à la* Woodford (2010) i.e the government recognizes that agents' beliefs about the distribution of shocks may be different from its own and wants to implement a policy robust to this unknown difference. As it is usual with robustness, the government objective is based on the assumption that the worst possible beliefs will always realize. Thus, the government solves a max-min problem. When the uncertainty is high enough, I show that, when given the freedom to select any contingent policy, the initial government selects a policy equal to the one implemented with optimal No Regret reforms. As a direct consequence, the welfare gap between the optimal contingent policy and the optimal No Regret reforms is fully closed. The intuition for this result is the following. No insurance may be provided by contingent bonds because, according to the worst beliefs, households believe low returns are more likely and so buy these bonds at low prices. As the uncertainty increases, the insurance becomes less and less attractive for the government. Without insurance, the initial government cannot do better than a government using No Regret fiscal reforms.

Finally, extending the analysis to introduce wealth in the population, I show the existence and provide a partial characterization of No Regret fiscal reforms. Their main characteristic is that agents with similar skills (but various wealth levels) should see their expected utilities shift by the exact same amount when a No Regret reform is implemented. Thus, the No Regret rule prevents redistribution across agents with different wealth levels (but with same skill). This is another rationale for the Equal Sacrifice principle. Equal Sacrifice principle keeps incentives to save unchanged. Adding skill heterogeneity, I use a mechanism design approach and show that the No Regret rule provides flexibility to cope with shocks to preferences regarding spending shocks and redistribution.

and Seppälä (2002) allow for 1 period non contingent bonds only, they also find that the government accumulates assets, and in the limit, the allocation is first-best and taxes are zero. Farhi (2010) adds capital and finds similar result when capital taxes are known 1 period in advance.

Related Literature

The rules versus discretion debate has been pervasive ever since Kydland and Prescott (1977). I am contributing to the fiscal part of this debate by introducing No Regret fiscal reforms which leave some discretion to future governments. Close to my work, Athey, Atkeson, and Kehoe (2005) model a present biased benevolent government which uses surplus and deficit to spend a fixed stream of government revenues. As I have, the usefulness of public spending is stochastic but my tax revenues are endogenous. They find optimal fiscal rules that restrict the amount a government may spend. Their time inconsistency issue comes from government's preferences whereas mine comes from the temptation to tax sluggish savings. Halac and Yared (2014) add persistence in the government's taste shocks, which I also model.

Optimal contingent policies in models close to mine were studied by Zhu (1992) and Chari, Christiano, and Kehoe (1994) which introduce capital in the Lucas Jr and Stokey (1983) seminal setup. Their government expenditures are exogenous but stochastic whereas I have that the *tastes* for government expenditures are exogenous but stochastic. They both study contingent policies under commitment.

Some researchers acknowledge that contingent policies are unrealistic and took steps towards more realistic assumptions by studying small departures from fully contingent policies with or without capital. However, even though their setups are more realistic, they still assume all decisions are taken by the initial government. Angeletos (2002) and Buera and Nicolini (2004) show that, with commitment, contingent bonds are not necessary since non-contingent bonds may play the same insurance role with a well chosen maturity structure. Aiyagari et al. (2002) restrict government bonds to be 1 period non-contingent bonds. Their bonds are risk-free in the sense that their returns are non-contingent. My bonds are risk free in the sense that households value its net of tax payoffs similarly in each state of the world.[2] In both cases bonds cannot provide insurance against shocks to public funds value. It is followed by Farhi (2010) which adds capital and restricts capital taxes to be known one period in advance. My capital tax must be such that households value net returns similarly in each state of the world. In both cases capital tax cannot provide insurance against shocks to public funds value. Gervais and Mennuni (2015) allows current investment to be productive right away, so capital is no more sluggish and taxing capital becomes distortionnary. All these papers still use contingent policies defined once and for all, which is the very feature I am avoiding here.

Closer to my work, departures from commitment to a contingent policy have been studied. Klein and Ríos-Rull (2003) studies a very realistic departure. Each successive government sets the current labour tax and the next period capital tax. Surprisingly, this is not enough to prevent high taxes on capital. Using a model without commit-

[2] Net returns are lower in state of the world in which households value more consumption because they consume less

ment (nor capital), Lucas Jr and Stokey (1983) and Debortoli, Nunes, and Yared (2017) show that governments are tempted to manipulate the expected values of their liabilities (assets) through high (low) future labour taxation which leads to low (high) bonds prices (government is a monopolist on the "store of value" market). Debortoli and Nunes (2013) adds endogenous government spending, which governments use to manipulate the values of their liabilities (assets) through high (low) low current spending which leads to high (low) current consumption and to low (high) bonds prices. They show that the government net asset position converges towards zero. This temptation to manipulate prices is absent in my work since, whatever policy is chosen, households should not regret their decisions so their expectations should be validated ex post. Lucas Jr and Stokey (1983) shows that contingent bonds allows a government without commitment ability to behave as if it could fully commit and to get insured against shocks. Without commitment nor contingent bonds Debortoli et al. (2017) show that, to reduce liability/asset manipulation and the associated costs, governments should favour flat maturity structures even though they don't provide much insurance. In my setup, bonds of any maturity have the same returns so the maturity structure is irrelevant. Debortoli and Nunes (2010) study loose commitment. They allow governments to renege at each period with a (possibly very small) positive probability. Without contingent bonds but with default, governments bonds can provide some insurance against bad shocks, recent work involves Dovis (2019). My government may reform its policy but never reneges on the No Regret rule nor defaults.

I contribute to the large literature on optimal capital taxation. Classic results (Atkinson & Stiglitz, 1976; Chamley, 1986; Judd, 1985) advocate for zero capital taxation, at least in the long run. My model has close to zero average capital taxation as in Zhu (1992), Chari et al. (1994), and Kocherlakota (2005)[3].

I study departures from common knowledge. It follows Sleet (2004) in which shocks are privately observed by governments. This feature adds a constraint on the implementable allocations because governments should always have incentives to correctly report the shocks. In my setup neither bonds nor capital tax provide insurance against shocks so governments do not have any incentive to misreport the shocks to wrongfully benefit from insurance and it does not matter whether shocks are private or pub-

[3]Recent research have been challenging their results (Straub & Werning, 2020). Apart from this debate, there are many known motives for positive capital taxes or subsidies : precautionary savings under income risk and financial constraints (Aiyagari, 1995), richer agents with different tastes (Saez, 2002), different work elasticities across ages (Erosa & Gervais, 2002), inequality-induced political instability (Farhi, Sleet, Werning, & Yeltekin, 2012), joy of giving in dynasties, preference for wealth (Saez & Stantcheva, 2018), government more patient than agents (Farhi & Werning, 2007), government caring for future generations (Farhi & Werning, 2010; Piketty & Saez, 2013), agents able to switch their labour income to capital income (Reis, 2011; Smith, Yagan, Zidar, & Zwick, 2019), firms' financial frictions (Abo-Zaid, 2014), imperfect competition in the goods markets (Guo & Lansing, 1999; Judd, 2002), heterogeneity in entrepreneurs' returns (Boar & Knowles, 2018) or in savers' returns (Gerritsen, Jacobs, Rusu, & Spiritus, 2020). My model carefully avoids all these motives.

lic.

My paper introduces Near-Rational Expectations into fiscal policy. The seminal papers are Woodford (2006) and Woodford (2010) but they study monetary policy. Recent papers on robust fiscal policy are Svec (2012) and Karantounias (2013). Just like I do, they consider that households' beliefs about shocks are uncertain. However, they consider the worst beliefs from the households' points of view whereas I consider the worst beliefs from the governments' point of view. Their governments are facing households with uncertain and pessimistic beliefs, my government is uncertain and pessimistic about households' beliefs. In Svec (2012) government's objective is to maximize households' welfare computed with households' pessimistic beliefs whereas in Karantounias (2013) it is to maximize households' welfare computed with governments' beliefs. The first has a political government, the second has a paternalistic government. In that sense, my government is paternalistic.

The notion of Regret has been introduced in the theory of decision under uncertainty by Savage (1951). A decision maker chooses between several actions. Each action has a payoff profile i.e. it yields different payoffs depending on the state of the world. For each state of the world, there is a best ex post action. The regret associated to an action and to a state of the world is the utility gain that could have been achieved by choosing the best ex post action. A decision maker who follows the Savage *minimax regret rule* chooses the action which minimizes the maximum regret over all states of the world. In their regret theory, Loomes and Sugden (1982) assume that decision makers choosing among two actions have a utility penalty (boost) when, for the realized state of the world, the non-chosen action would have yielded a higher (lower) payoff that they call regret (rejoicing). Decision makers anticipate regret and rejoicing when choosing their actions. My paper follows Savage (1951) and defines regret as the difference between the utility level that could have been obtained had a household known the realized path of the economy in advance and the actual utility level she got. However, households are expected utility maximizers and neither the minimax regret rule nor the regret theory is used to model their decision making processes.

The paper is organized as follows. Section 2 describes the model. Sections 3 introduces the main novelty of this paper, namely No Regret fiscal reforms and the No Regret rule. Section 4 studies Near-Rational Expectations. Section 5 introduces wealth and skill heterogeneity. Section 6 contains numerical illustrations. The last section concludes.

2 The setup

2.1 Environment

The economic environment is a neo-classical growth model with endogenous government expenditures. Preferences regarding government expenditures are stochastic. It is the only source of uncertainty. These shocks are denoted $\{\theta_t\}_{t=0}^{\infty}$. Time is discrete and the horizon is infinite with periods denoted t.

Any infinite sequence $\{X_t\}_{t=0}^{\infty}$ will be denoted X and $\{X_s\}_{s=0}^{t}$ will be denoted X^t. When this sequence depends on shocks θ, $\{X_t(\theta^t)\}_{t=0}^{\infty}$ will be denoted $X(\theta)$ and $\{X_s(\theta^s)\}_{s=0}^{t}$ will be denoted $X^t(\theta^t)$. When no confusion is possible these two sequences will be respectively denoted X and X^t.

Since θ is the only source of uncertainty, I write $\mathbb{E}_t[\,\cdot\,] \equiv \mathbb{E}[\,\cdot \mid \theta^t\,]$.[4] For simplicity, θ is a first-order Markov chain, it can take N different values and has a non decreasing[5] transition probability matrix M such that the probability distribution of θ_t, always converges towards an invariant distribution. At each time t, θ_t is publicly observable and the probability distribution of θ is common knowledge at time 0. I relax this last assumption later.

Technology. The economy is closed and the production technology is represented by a neo-classical production function $F(\cdot,\cdot)$ with constant returns to scale in capital and labour. Output can be used either for consumption, government expenditures or capital productive at the next period. The ressource constraint of the economy at period t when θ^t realized is

$$C_t(\theta^t) + G_t(\theta^t) + K_{t+1}(\theta^t) \leq F(K_t(\theta^t), L_t(\theta^t))^{6} \tag{1}$$

At time t and when shocks θ^t realized, productive capital is denoted $K_t(\theta^{t-1})$, labour $L_t(\theta^t)$, consumption$C_t(\theta^t)$, government expenditures $G_t(\theta^t)$ and the next period capital $K_{t+1}(\theta^t)$. An *allocation* $\{C, L, G, K\}$ is *resource feasible* when (1) is met at all periods and for all realizations of θ. Perfect competition is assumed so the gross return on capital is equal to the marginal product of capital $F_K(K_t(\theta^t), L_t(\theta^t))$ and the gross wage is equal to the marginal product of labour $F_L(K_t(\theta^t), L_t(\theta^t))$. Perfect competition and constant return to scale in production function imply no profit, so firm ownership is irrelevant.

Households. There are an infinite number of identical households. The households' preferences are given by

[4] I use the σ-algebra generated by θ as filtration

[5] i.e. if $\theta' \geq \theta$, then for all x, $\mathbb{P}[\theta_{t+1} \geq x \mid \theta_t = \theta'] \;\geq\; \mathbb{P}[\theta_{t+1} \geq x \mid \theta_t = \theta]$

[6] Without loss of generality, F is gross of depreciated capital

$$\mathbb{E}_0[\sum_{t=0}^{\infty}\beta^t\ [\ u(C_t(\theta^t)) - v(L_t(\theta^t)) + \theta_t w(G_t(\theta^t))\ ]\ ]$$

Where $C_t(\theta^t)$, $L_t(\theta^t)$ and $G_t(\theta^t)$ are consumption, labour and government expenditures at time t when shocks θ^t realized. $\beta \in (0,1)$ is the discount factor. I assume that the functions $u(\cdot)$ and $w(\cdot)$ are strictly increasing and strictly concave and the function $v(\cdot)$ is strictly increasing and strictly convex.

The representative household can invest in capital and can buy (or sell) 1 period risk free government bonds. She takes gross wages $F_L(K(\theta), L(\theta))$, gross returns on capital $F_K(K(\theta), L(\theta))$, bond prices $P(\theta)$ and transfers $T(\theta)$ as given. She also takes linear taxes on labour $\tau^L(\theta)$, on capital $\tau^K(\theta)$ and on government bonds $\tau^B(\theta)$ as given.

I denote net of tax wages $W(\theta) \equiv (1-\tau^L(\theta))F_L(K(\theta), L(\theta))$ and net of tax capital returns $R^K(\theta) \equiv (1-\tau^K(\theta))F_K(K(\theta), L(\theta))$ [7]. At period t when θ^t realized, the households use their net labour incomes $W_t(\theta^t)L_t(\theta^t)$, their non-negative lump-sum transfers $T_t(\theta^t)$, their capital and bonds net holdings $R_t^K(\theta^t)K_t(\theta^{t-1}) + (1-\tau_t^B(\theta^t))B_t(\theta^{t-1})$ to buy consumption goods $C_t(\theta^t)$ and to buy new capital $K_+1(\theta^t)$ and bonds $B_{t+1}(\theta^t)$ at price $P_t(\theta^t)$. Thus, the household's budget constraint at period t when θ^t realized is

$$\begin{aligned} C_t(\theta^t) + K_{t+1}(\theta^t) + P_t(\theta^t)B_{t+1}(\theta^t) \\ = R_t^K(\theta^t)K_t(\theta^{t-1}) + (1-\tau_t^B(\theta^t))B_t(\theta^{t-1}) + W_t(\theta^t)L_t(\theta^t) + T_t(\theta^t) \end{aligned} \quad (2)$$

To rule out explosive debt for the household and for the government, there are bounds M^- and M^+ to the representative household's net asset position so that at period t and realization θ^t

$$M^- \leq K_{t+1}(\theta^t) + P_t(\theta^t)B_{t+1}(\theta^t) \leq M^+ \quad (3)$$

Optimal saving using capital investment at period t gives the following Euler Equation

$$u'(C_t(\theta^t)) = \mathbb{E}_t[\ \beta R_{t+1}^K(\theta^{t+1})u'(C_{t+1}(\theta^{t+1}))\ ] \quad (4)$$

Optimal saving using bonds at period t gives another Euler Equation

$$u'(C_t(\theta^t)) = \mathbb{E}_t[\ \beta R_{t+1}^B(\theta^{t+1})u'(C_{t+1}(\theta^{t+1}))\ ] \quad (5)$$

[7] Capital *and* capital incomes are taxed at rate τ^K. A tax on capital incomes *only* with a rate $\frac{\tau^K F_K(K(\theta),L(\theta))}{F_K(K(\theta),L(\theta))-1}$ is exactly equivalent

Where $R^B_{t+1}(\theta^{t+1}) \equiv (1-\tau^B_{t+1}(\theta^{t+1}))/P_t(\theta^t)$ is the the net of tax return on bonds. Optimal labour effort any period t implies

$$v'(L_t(\theta^t)) = W_t(\theta^t)u'(C_t(\theta^t)) \tag{6}$$

Proofs are in the Appendix

Government. The government is benevolent so its preferences are identical to the households'. Government directly sets its expenditures.[8]

To fund its stream of expenditures $G(\theta)$ and lump-sum positive transfers $T(\theta)$, the government raises linear taxes on labour incomes $\tau^L(\theta)$, on the households' capital assets (including gross returns) $\tau^K(\theta)$, and on maturing bonds $\tau^B(\theta)$. At each period t, the government may also sell a quantity $B_{t+1}(\theta^t)$ of one-period bond at price $P_t(\theta^t)$ and must pay back the maturing bonds $B_t(\theta^{t-1})$. When $B_{t+1}(\theta^t)$ is negative the government holds claims on households. The government's budget constraint at each time t is

$$G_t + T_t + B_t = \tau^L_t F_L(K_t, L_t)L_t + \tau^K_t F_K(K_t, L_t)K_t + \tau^B_t B_t + B_{t+1}P_t \tag{7}$$

Competitive Equilibrium. I may now define my equilibrium concept:

Definition 1 ***(Competitive Equilibrium)***
A ***competitive equilibrium*** *is*

1. *A contingent allocation* $\{C_t, L_t, G_t, K_{t+1}\}_{t=0}^{\infty}$
2. *A contingent policy* $\{G_t, T_t, \tau^L_t, \tau^K_t, \tau^B_t, B_{t+1}\}_{t=0}^{\infty}$
3. *Contingent prices* $\{W_t, R^K_t, P_t\}_{t=0}^{\infty}$

such that:

1. $W_t = (1-\tau^L_t)F_L(K_t, L_t)$ *and* $R^K_t = (1-\tau^K_t)F_K(K_t, L_t)$ *for all* t

[8]The usual assumption is that public expenditures are exogenous but my assumption is arguably more realistic. Furthermore, there are some time inconsistency problem with endogenous government spending which my approach also solves. Debortoli and Nunes (2013) gets significantly different results than Krusell, Martin, Ríos-Rull, et al. (2006) by assuming endogenous spending

2. *Household's decisions are optimal*

3. *Resources constraints are met*[9]

An allocation is called *implementable* when there exists a policy and prices such that the triplet forms a competitive equilibrium. Similarly, a policy is called *implementable* when there exists an allocation and prices such that they form a competitive equilibrium. I also define a *t-equilibrium* when the allocation, the policy and the prices start at time t. I then say that allocations and policies are *t-implementable.* Note that households are rational, so the policy anticipated by the households realizes for all path θ, they are no surprise or unanticipated changes in the policy.

3 No Regret Fiscal Reforms

With contingent policies, the initial government fully binds all future governments. The opposite approach is full discretion, no government may bind any of its successors. This paper's approach is halfway between these two extremes: each government may *partially bind* future governments so that each government has *some discretion* left. It works as follows: the initial government chooses an implementable *non-contingent* policy and each successive government may reform this policy subject to the No Regret rule. These reforms are called No Regret fiscal reforms.

3.1 No Regret fiscal reforms

First, let us define a *fiscal reform*.

Definition 2 ***(Fiscal Reform)***
A fiscal reform at time t_R is a change of policy, so that the current non-contingent policy $\{G_s, T_s, \tau_s^L, \tau_s^K, \tau_s^B, B_s\}_{\{s\geq t_R\}}$ is replaced by a new non-contingent policy $\{\tilde{G}_s, \tilde{T}_s, \tilde{\tau}_s^L, \tilde{\tau}_s^K, \tilde{\tau}_s^B, \tilde{B}_s\}_{\{s\geq t_R\}}$

At each period t, the timing is the following. First, θ_t is observed. Then the government announces a fiscal reform if it wishes to. Finally, households take their decisions. In this paper I impose that fiscal reforms are subject to the No Regret rule. I then say that the government is under the No Regret rule.

Definition 3 ***(No Regret Rule)***
Under the No Regret Rule, a fiscal reform at time t_R must be such that:

[9] Since the economy is closed, the ressource constraint (1), and the households' budget constraint (2) implies by Walras' law that the governement's budget constraint (7) holds as well.

1. *The new non-contingent policy is t_R-implementable*

2. *The optimal households' decisions before t_R are identical*

   – *Under the old policy*

   – *Under the new policy*

Such fiscal reforms are called *No Regret Fiscal Reforms*. Of course rational households may (and will) anticipate fiscal reforms. However, as the next lemma shows, it is not useful for households to consider possible future reforms.

Lemma 1 ***(Equilibrium under the No Regret rule)***
Under the No Regret rule, it is optimal for households to take decisions as if it were common knowledge that the current policy will never be reformed.

The intuition for this result is as follows. If all households behave as if the current policy will never change then the best strategy for a given atomistic household i is also to behave as if the policy plan will never change. Indeed, if the current non-contingent policy plan is never reformed it is by definition the best strategy. If the policy is reformed, reforms will be No Regret so i won't regret this strategy. In other words, the best strategy for households is to behave as if the current non-contingent policy will never be changed and to (rightfully) trust the government to never make them regret this strategy. The No Regret rule implies that, whatever the reforms announced by the government, households decisions are always *ex post* optimal.

3.2 Ex Post Euler Equation

Under the No Regret rule, households never regret their past decisions, whether the policy is reformed or not. This implies that at time t for the realization θ^t, households don't regret their $(t-1)$-saving decisions $K_t(\theta^{t-1})$ and $B^t(\theta^{t-1})$. This implies

$$u'(C_{t-1}(\theta^{t-1})) = \beta R_t^K(\theta^t) u'(C_t(\theta^t))$$

$$u'(C_{t-1}(\theta^{t-1})) = \beta R_t^B(\theta^t) u'(C_t(\theta^t))$$

Bonds and capital must have the same *ex post* net returns which I denote R_t. I thus get the following *ex post* Euler Equation which always holds under the No Regret rule.

$$u'(C_{t-1}(\theta^{t-1})) = \beta R_t(\theta^t) u'(C_t(\theta^t)) \tag{8}$$

Four things are worth mentionning. First, the *ex post* Euler Equation (8) is stronger than the two Euler equations for capital (4) and bonds (5). Second, since $R^B = R^K$, households are indifferent between government bonds and capital. Thus the compositions of their portfolios are indeterminate and only their total wealth matters. Third, since capital and bonds have the same ex post returns it is useless for the government to trade capital on top of its own bonds. Capital won't help complete the market nor provide any insurance to the government contrary to the case with *untaxed risk free* bonds studied by Farhi (2010). Fourth, considering only 1 period bonds is without loss of generality under the No Regret rule since all bonds should yield the same net of tax returns.

3.3 Relaxed government problem

At each time t, the problem of the government is to find the optimal No Regret reform. This reform may change current and future prices and so will *indirectly* lead households to take current decisions $\{ C_t, L_t, K_{t+1}, B_{t+1} \}$. In this subsection, I study a relaxed problem where the government *directly* chooses current prices and current households' decisions. This *direct* choice is subject to five constraints which always hold under the No Regret rule: the resource constraint (1), the households budget constraint (2), the optimal labour equation (6), the ex post Euler Equation (8) and the debt limits (3). The time t government wants to maximize its utility $V_t(\cdot)$ which is the sum of its *current* flow utility and of all the (discounted) flow utilities of future governments. So, its utility $V_t(\cdot)$ is the sum of its *current* flow utility and of the (discounted) utility of the next period government $\beta V_{t+1}(\cdot)$. Hence the problem has the following recursive form

$$
\begin{aligned}
V_t(C_{t-1}, K_t, B_t, \theta_t) = \max_{\substack{C_t, L_t, G_t, K_{t+1} \\ R_t, W_t, B_{t+1}, T_t}} \; & u(C_t) - v(L_t) + \theta_t w(G_t) \\
& + \beta \mathbb{E}_t[\, V_{t+1}(C_t, L_t, K_{t+1}, B_{t+1}, \theta_{t+1})\,] \\
s.t. \quad & (1), (2), (3), (6), (8)
\end{aligned}
$$

This relaxed problem has three endogenous state variables and eight control variables. It can be rewritten with two endogenous state variables and five control variables. To do so, I define the households' total net asset position expressed in marginal utility

$$A_{t+1} \equiv u'(C_t)(K_{t+1} + P_t B_{t+1})/\beta$$

Using the fact that transfers are positive $T_t \geq 0$ and the stationarity of the relaxed problem, I can rewrite it as follows

$$
\begin{aligned}
V(K_t, A_t, \theta_t) = \quad \max_{C_t, L_t, G_t, K_{t+1}, A_{t+1}} \quad & u(C_t) - v(L_t) + \theta_t w(G_t) \; + \; \beta \mathbb{E}_t[\, V(K_{t+1}, A_{t+1}, \theta_{t+1}) \,] \\
s.t. \quad & C_t + G_t + K_{t+1} \leq F(K_t, L_t) \qquad (\lambda_t) \\
to \quad & A_t \leq u'(C_t) C_t - v'(L_t) L_t + \beta A_{t+1} \qquad (\gamma_t) \\
to \quad & M^- \leq \frac{\beta A_t}{u'(C_t)} \leq M^+
\end{aligned}
$$

The first inequality represents the ressource constraint and the second is called the implementabilty constraint. The associated Lagrange multipliers are λ_t and γ_t.

Time 0 problem. At time 0, the initial government may choose any implementable non-contingent policy. As it is well known, the government would like to heavily tax accumulated wealth to finance its future expenditures without having to distort the economy. To model a limit to initial wealth taxation, I assume that the initial households' wealth A_0 is exogenous.

Time t problem. When $\gamma_t = 0$, the constraint is slack and only the ressource constraint binds i.e. it is the no tax first best problem. In other words, the government is wealthy enough to pay for the first best expenditures without any distortive taxation. When $\gamma_t > 0$, the government must tax households in order to finance its expenditures or its debt. The higher the usefulness of expenditures or the higher its debt, the higher the γ_t. The first order condition with respect to A_{t+1} and the envelope condition with respect to A_t gives for all t

$$\gamma_t = \mathbb{E}_t[\gamma_{t+1}] \geq 0$$

Hence γ is a positive super-martingale and therefore must converge almost surely towards a random variable denoted $\overline{\gamma}$ (Doob's first martingale convergence theorem). Furthermore the higher the θ_t, the higher future θ_{t+s} ,[10] and so the higher the need to tax and the higher the multiplier of the implementability constraint γ_t. So, for a given $\gamma_{t-1} > 0$, the value of γ_t is strictly increasing with the realization of θ_t. This means that γ_t cannot be constant across different shocks θ_t unless γ_t is zero. But since $\{\gamma_t\}_t$ converges almost surely, it must converge towards zero. The intuition for this result is that government accumulates assets to self insure until it has enough asset to be able pay for the first best expenditures level without taxation.

For all times t, successive governments face similar problems taking as given K_t and A_t. As the following theorem summarizes, there is a primal and sequential approach to the relaxed problem.

[10] If $\theta_t \geq \tilde{\theta}_t$, then the random variable $\{\theta_{t+s}\}_{s>0}$ conditional on θ_t has first-order stochastic dominance over $\{\tilde{\theta}_{t+s}\}_{s>0}$ conditional on $\tilde{\theta}_t$. This is because the transition probability of θ is non decreasing

Theorem 1 ***(No Regret Primal Approach)***
Under the No Regret rule, at each time t and for any realization θ^t the government solves:

$$\max_{\substack{\{C_s(\theta^s),L_s(\theta^s),\\ G_s(\theta^s),K_{s+1}(\theta^s)\}_{s\geq t}}} \mathbb{E}_t[\sum_{s=t}^{\infty} \beta^s\ [\ u(C_s(\theta^s)) - v(L_s(\theta^s)) + \theta_s w(G_s(\theta^s))\ ]\ ]$$

$$subject\ to\quad C_s(\theta^s) + G_s(\theta^s) + K_{s+1}(\theta^s) \leq F(K_s(\theta^{s-1}), L_s(\theta^s))\ \ for\ all\ s$$

$$to\quad A_t(\theta^{t-1}) \leq \sum_{s=t}^{\infty} \beta^s\ [\ u'(C_s(\theta^s))C_s(\theta^s) - v'(L_s(\theta^s))L_s(\theta^s)\ ]\ \ almost\ surely$$

$$and\ to\quad M^- \ \leq\ \frac{\beta A_s(\theta^{s-1})}{u'(C_s(\theta^s))}\ \leq\ M^+$$

(**Proof:** Appendix) Since the economy reaches first best, it is useful to consider the next lemma.

Lemma 2 ***(Uniformly bounded first best allocation)***
The first best allocation is such that C, L, G, K are uniformly bounded from above and away from 0.

Remember that the debt limit (3) states that $\frac{A_t}{u'(C_t)}$ is bounded. Since the optimal solution of the relaxed problem reaches the first best and thanks to lemma 2, $\limsup_{t\to\infty}\ u'(C_t)\Gamma isuniformly$
$\beta^t A_t\}_t$ converges almost surely towards zero. The allows to sum the implementablity constraints at all times t to get that the optimal solution of the relaxed problem is such that

$$A_0\ \leq\ \sum_{t=0}^{\infty} \beta^t\ [\ u'(C_t)C_t - v'(L_t)L_t\ ]\ \ almost\ surely$$

Tax policy. To implement the allocation which solves the previous problem, the time t government may announce the following non-contingent policy for all $s \geq t$:

- $G_s = G_s(\theta^s)$
- $\tau_s^L = 1 - \frac{v'(L_s(\theta^s))}{F_L(K_s(\theta^{s-1}),L_s(\theta^s))u'(C_s(\theta^s))}$
- $\tau_s^K = 1 - \frac{u'(C_{s-1}(\theta^{s-1}))}{\beta F_K(K_s(\theta^{s-1}),L_s(\theta^s))u'(C_s(\theta^s))}$
- $\tau_t^B = 1 - \frac{P_{t-1}(\theta^{t-1})u'(C_{t-1}(\theta^{t-1}))}{\beta u'(C_t(\theta^t))}$
- $\tau_s^B = 0$ for $s > t$

Where θ^s can be any continuation of the realized shocks θ^t. Taxes on labour are set such that the optimal labour equation (6) holds. Taxes on capital and bonds are set such that the ex post Euler Equation (8) holds. These two equations and the implementability condition implies that the budget constraint of the households hold and that households' transversality condition also holds. These imply that households' decisions are optimal. (**Proof:** Appendix) and that the solution of the relaxed problem is the solution of the original problem.

Tax Indeterminacy. The previous non-contingent policy announced at time t_R is the policy that will take place if θ is realized. If another $\tilde{\theta}$ realizes then the government will announce another policy. This means that, even though the *actual* realized tax policy is uniquely defined for each realization θ, the optimal No Regret reform announced at time t when a given θ^t realizes is not uniquely defined.

Contingent policy. Let us make some comparisons with the initial government setting the optimal contingent policy. At time $t = 0$, this problem is to be compared to a similar classic result obtained for the optimal contingent allocation:

Theorem 2 ***(Contingent Primal Approach)***
The optimal contingent allocation solves the following sequential problem:

$$\max_{\substack{\{C_s(\theta^s),L_s(\theta^s),\\ G_s(\theta^s),K_{s+1}(\theta^s)\}_{s\geq 0}}} \quad \mathbb{E}_0[\sum_{s=0}^{\infty} \beta^s \,[\, u(C_s(\theta^s)) - v(L_s(\theta^s)) + \theta_s w(G_s(\theta^s)) \,]\,]$$

$$subject\ to \quad C_s(\theta^s) + G_s(\theta^s) + K_{s+1}(\theta^s) \leq F(K_s(\theta^{s-1}), L_s(\theta^s)) \quad for\ all\ s$$

$$to \quad A_0 \leq \mathbb{E}_0[\sum_{s=0}^{\infty} \beta^s \,[\, u'(C_s(\theta^s))C_s(\theta^s) - v'(L_s(\theta^s))L_s(\theta^s) \,]\,]$$

$$and\ to \qquad M^- \;\leq\; \frac{\beta A_s(\theta^{s-1})}{u'(C_s(\theta^s))} \;\leq\; M^+$$

These two problems are very close. The difference lies in the fact that for the second problem the implementability inequality must be met in expectation whereas for the first it should hold for all realizations of the shocks θ. This directly implies that the optimal contingent policy achieves a higher welfare level than the optimal No Regret policy. Numerical simulations later show that this difference is very small. This also implies that, when there is no shock, both problems have the same solution. Under the No Regret rule, the initial government may announce a non-contingent plan which is not reformed in the absence of shocks. This means that the No Regret rule, if respected, prevents time inconsistency problems such as capital taxation, default on government debt, bonds' prices manipulation through labour taxes and government spending.

3.4 Other properties

No Regret fiscal reforms and contingent bonds. We have seen that, in order to avoid any regret, bonds and capital should always yield the same net returns. However, an *alternative* No Regret rule could be 1) contingent bonds are allowed and must always be paid back without any tax, 2) capital and labour taxation should be set such that households' don't regret their previous decisions other than their investments in bonds. This alternative No Regret rule allows to use contingent bonds. Under this alternative No Regret rule, the government may fully use contingent bonds and implement the optimal contingent allocation without having to rely on contingent planning at $t = 0$.

Anticipation. When a No Regret reform is announced, by definition, households don't regret their past behaviours. This implies that, had they known about the reform in advance, they wouldn't have behaved differently. The reverse also holds, if a household is informed about a reform in advance and yet doesn't modify its actions until the reform is implemented, then this reform must be a No Regret fiscal reform. One gets the following lemma:

Lemma 3 ***(No Regret reform characterization)***
For a given fiscal reform at t_R, these two points are equivalent:

1. *The reform is No Regret*
2. *Any household i behaves identically before t_R, whatever the time $t_R^i \leq t_R$, she is informed about the reform.*

Proof: See Appendix

Private information. An another feature of No Regret reforms is that under the No Regret rule, it doesn't matter whether the shocks are governments' private information or public information.

Lemma 4 ***(Private Shocks θ)***
*If θ is **privately observed** by the government, then, the optimal No Regret policy and the resulting allocation are unchanged.*

One can even show that households could believe anything about the shocks. When the government is under the No Regret rule, their beliefs have no effect on their decisions.
Proof: See Appendix

4 Near-Rational Expectations

In this section, I study robustness to Near-Rational Expectations à la Woodford (2010). The government recognizes that households' expectations about shocks are not necessary equal to its own expectations. The government is *uncertain* about the households' beliefs and wants to implement a policy *robust* to this uncertainty. I establish that if the robustness requirement is large enough, the optimal allocation is implementable under the No Regret rule. In other words, even if commitment to a full contingent policy is available, the government is not worst off under the No Regret rule.

4.1 The robust problem

Households' beliefs about the probability distribution of shocks θ are described by a probability distribution denoted p^B, it is not necessary equal to the probability distribution used by the government denoted p. There is a set of probability distributions $\mathcal{P}$ such that the government wishes to choose a policy that will be as good as possible in the case of any beliefs p^B in the set $\mathcal{P}$. The larger the set $\mathcal{P}$, the larger the robustness required by the government. I assume that the set $\mathcal{P}$ is a closed convex set such that the probability distribution used by the government p is interior.

In this section, I assume that commitment to a full contingent policy is possible. Since households' beliefs p^B are unknown, the government cannot predict households' decisions and the corresponding tax revenues. So instead of writing a contingent policy which is a function of time t and the realization of shocks θ^t, it must write a contingent policy which is also function of past households' decisions. The timing is the following. At t=0, the government announces its policy such that, government expenditures, labour tax, capital tax, and next period contingent returns on bonds at time t are functions of the realization θ^t and possibly of all households' decisions taken before t (i.e. $C^{t-1}, L^{t-1}, K^t, B^t$). The price of contingent bonds P_t may vary depending on households' beliefs, so the quantity sold B_{t+1} is mechanically set such that the government is able to pay for its current expenditures G_t. Then, the households' beliefs p^B are chosen in the set $\mathcal{P}$ to *minimize* the expected utility computed with the government's expectations p.

Information structure. I assume that households all have the same beliefs p^B and that p^B is common knowledge among them. Knowing the contingent policy and the beliefs p^B, households are able to predict all variables for any realization of θ.[11]

[11] An alternative assumption would be to assume that the government commits to a plan of contingent net prices independent of households' past decisions. Each atomistic household does not need to compute the others' decisions to take the optimal decisions for herself.

The max-min problem. The government is *benevolent* and *paternalistic* in the sense that its objective is to *maximize* the expected utility computed with its own expectations p. The problem of the government is the following:

$$\begin{aligned}
\max_{Policy} \quad \min_{p^B \in \mathcal{P}} \quad & \mathbb{E}_0[\sum_{s=0}^{\infty} \beta^t \, [\, u(C_t(\theta^t,p^B)) - v(L_t(\theta^t,p^B)) + \theta_t w(G_t(\theta^t,p^B))\,]\,] \\
& s.t. \;\; C_t(\theta^t,p^B) + G_t(\theta^t,p^B)) + K_{t+1}(\theta^t,p^B)) \le F(K_t(\theta^{t-1},p^B)), L_t(\theta^t,p^B)) \\
& to \;\;\; \{C_t(\theta^t,p^B), L_t(\theta^t,p^B), K_{t+1}(\theta^t,p^B), B_{t+1}(\theta^t,p^B)\} \; optimal \; decisions \; with \; beliefs \;\; p^B \\
& to \;\;\; M^- \le K_{t+1}(\theta^t,p^B) + B_{t+1}(\theta^t,p^B) \le M^+
\end{aligned}$$

Theorem 3 ***(Robust primal sequential approach)***
The solution to the max-min problem solves the following problem

$$\begin{aligned}
\max_{\substack{\{C_t(\theta^t),L_t(\theta^t),\\ G_t(\theta^t),K_{t+1}(\theta^t)\}_{t\ge 0}}} \quad & \mathbb{E}_0[\sum_{t=0}^{\infty} \beta^t \, [\, u(C_t(\theta^t)) - v(L_t(\theta^t)) + \theta_t w(G_t(\theta^t))\,]\,] \\
subject \; to \;\; & C_t(\theta^t) + G_t(\theta^t) + K_{s+1}(\theta^t) \le F(K_t(\theta^{t-1}), L_t(\theta^t)) \;\; for \; all \; \theta^t \\
to \;\; A_0 \le \min_{p^B \, \in \, \mathcal{P}} \; & \mathbb{E}_0[\sum_{t=0}^{\infty} \beta^t \, [\, u'(C_t(\theta^t))C_t(\theta^t) - v'(L_t(\theta^t))L_t(\theta^t)\,]\, \frac{p^B(\theta^t)}{p(\theta^t)} \,] \\
and \; to \quad & M^- \;\le\; K_t(\theta^{t-1}) + B_t(\theta^{t-1}) \;\le\; M^+
\end{aligned}$$

(Proof: Appendix)

I denote $\{C_t^\star(\theta^t), L_t^\star(\theta^t), G_t^\star(\theta^t), K_{t+1}^\star(\theta^t)\}_{t=0}^{\infty}$ the contingent allocation which solves the maximization problem and $p^\star$ the beliefs which solve the minimization problem.

Tax policy. I now turn to the implementation of the allocation $\{C_t^\star(\theta^t), L_t^\star(\theta^t), G_t^\star(\theta^t), K_{t+1}^\star(\theta^t)\}_{t=0}^{\infty}$ with a contingent policy. I establish that the allocation is implementable for all beliefs $p^B \in \mathcal{P}$.

Lemma 5 ***(Robust optimal policy)***
For any possible beliefs $p_B \in \mathcal{P}$, it is possible to implement the allocation $\{C_t^\star(\theta^t), L_t^\star(\theta^t), G_t^\star(\theta^t), K_{t+1}^\star(\theta^t)\}_{t=0}^{\infty}$ with the contingent policy $\{G_t(\theta^t), \tau_t^L(\theta^t), \tau_t^K(\theta^t), R_t^B(\theta^t), T_t(\theta^t, B_t), \}_{t=0}^{\infty}$ defined as follows:

- $G_t(\theta^t) = G_t^\star(\theta^t)$
- $\tau_t^L(\theta^t) = 1 - \frac{v'(L_t^\star(\theta^t))}{F_L(K_t^\star(\theta^{t-1}), L_t^\star(\theta^t))u'(C_t^\star(\theta^t))}$
- $\tau_t^K(\theta^t) = 1 - \frac{u'(C_{t-1}^\star(\theta^{t-1}))}{\beta F_K(K_t^\star(\theta^{t-1}), L_t^\star(\theta^t))u'(C_t^\star(\theta^t))}$

- $R_t^B(\theta^t) = [\ \frac{A_t^\star}{u'(C_t^\star(\theta^t))} - \frac{u'(C_{t-1}^\star(\theta^{t-1}))KC_t^\star(\theta^t)}{u'(C_t^\star)\beta}\ ]\ /\ [\ \frac{\beta\mathbb{E}_{t-1}[\ A_t\ ]}{u'(C_{t-1})} - K_t\ ]$
- $T_t(\theta^t, B_t) = [\ A_t^\star - u'(C_{t-1}^\star(\theta^{t-1}))K_t^\star(\theta^{t-1})/\beta - B_t R_t^B(\theta^t)\ ]\ /\ [\ u'(C_t^\star(\theta^t))\ ] \quad \geq 0$

With $A_t^\star(\theta^t) \equiv \mathbb{E}_t[\ \sum_{s=0}^{\infty} \beta^{s-t}\ [\ u'(C_s(\theta^s))C_s(\theta^s) - v'(L_s(\theta^s))L_s(\theta^s)\ ]\ \frac{p^\star(\theta^s|\theta^t)}{p(\theta^s|\theta^t)}\ ]$

The *worst* households' beliefs are the beliefs which minimize the price of bonds, so any other beliefs yield higher prices and lower quantities of bonds sold. Since the government sells less bonds, positive lump-sum transfers are enough to make sure that households have the same wealth whatever their beliefs are.[12]

Break-even beliefs. As mentioned above, the larger the set $\mathcal{P}$, the larger the robustness required by the government. This subsection establishes that there are beliefs denoted p^{BE} such that, if $p^{BE} \in \mathcal{P}$, then the optimal robust allocation can be implemented under the No Regret rule.

Let us denote the optimal contingent allocation under the No Regret rule, denoted $\{C_t^{NR}(\theta^t), L_t^{NR}(\theta^t), G_t^{NR}(\theta^t), K_{t+1}^{NR}(\theta^t)\}_{t=0}^{\infty}$ is associated with Largrange multipliers for the implementability constraints $\{\gamma_t^{NR}(\theta^t)\}_{t=0}^{\infty}$. Using the First Order conditions and envelope theorems gives that for any realization θ^t

$$\gamma_t^{NR}(\theta^t) = \frac{\theta_t w'(G_t^{NR}(\theta^t)) - v'(L_t^{NR}(\theta^t))}{v''(L_t^{NR}(\theta^t))L_t^{NR}(\theta^t) - v'(L_t^{NR}(\theta^t))}$$

These multipliers $\{\gamma_t^{NR}(\theta^t)\}_{t=0}^{\infty}$ represent the shadow prices of the values of household's wealth $A_t^{NR}(\theta^{t-1})$. Under the No Regret rule, it is not possible to transfer households' wealth from realizations of shocks θ^t where its shadow value $\gamma_t^{NR}(\theta^t)$ is high to realizations of shocks where it is low. These transfers can be realized with contingent bonds. But the prices of these bonds may vary with beliefs p^B. The following beliefs which I call break-even beliefs and denote p^{BE} are built such that the prices of contingent bonds are low enough to bring at most zero marginal welfare gains. For any realization of shocks θ^t, the break-even beliefs p^{BE} have the following marginal probability distribution for the next period shock θ_{t+1}:

$$p_{t+1}^{BE}(\ \theta_{t+1} \mid \theta^t\ ) \equiv p_{t+1}(\ \theta_{t+1} \mid \theta^t\ ) \frac{\gamma_{t+1}^{NR}(\theta^{t+1})}{\mathbb{E}_t[\ \gamma_{t+1}^{NR}(\theta^{t+1})\ ]}$$

The break-even beliefs p^{BE} are deviations of the true distribution of shocks p. With the break-even beliefs p^{BE}, shocks which yields high shadow prices of the values of

[12]Of course, positive lump-sum transfers with positive taxation are not optimal so the government could achieve higher welfare with beliefs different than $p^\star$.

household's wealth are more likely. So using contingent bonds with low payoffs with these high shocks can only be sold at prices too low to be worth worth emitting.

This gives us the main result of this section:

Theorem 4 ***(Near-Rational expectations and No Regret rule)***
If the government robustness requirement are such that $p^{BE} \in \mathcal{P}$, then the optimal robust allocation can be implemented under the No Regret rule.

This result establishes that, when the concern of the government for robustness to Near-Rational Expectations is high enough, the government doesn't want to transfer households wealth from one state to another with contingent bonds. Numerical simulations show that the break-even beliefs p^{BE} are close to the true distribution p, so even a low concern for robustness to Near-Rational Expectations is enough to make contingent bonds useless.

5 Heterogeneity

In this section, I introduce heterogeneity among households. In the previous sections, I have studied how the No Regret rule tackled several classic time-inconsistency problems: capital taxation, default on bonds and bonds' prices manipulation. Without capital, nor government bonds in the economy, these time-inconsistency problems are absent and full discretion is equivalent to contingent policies. faces this time inconsistency problem.

5.1 Wealth heterogeneity

Let us first introduce wealth heterogeneity and assume that households still have the same preferences but may have different asset holdings or wealth a. The At each time t and for each household i, I define a_t^i as the wealth with which households i enters period t before production takes place. Under the No Regret rule, the composition of wealth is irrelevant since to all assets (capital, government bonds and households' debt) have identical net of tax returns. The wealth levels are observed by the government. In addition to linear taxes on labour and assets, the governments' non-contingent taxation plan involves a joint schedule such that household i with wealth and net returns $R_t a_t^i$ and with net wage $W_t L_t^i$ pays $\tau_t^{Joint}(R_t a_t^i, W_t L_t^i)$. Depending on her wealth $a_{i_{t_R}}$, any household i expects to get a total utility denoted $U(a_{t_R}^i)$ if the government's plan is never reformed. With a reform at time t_R, the new total expected utilities are denoted $\tilde{U}(a_{t_R}^i)$. The next results shows that it is sufficient and necessary for a reform to be a No Regret reform that it shifts households total expected utility distribution in parallel. In other words, all agents, whatever their wealth levels, should see their expected utilities change by the same amount.

Theorem 5 ***(No Regret reform with wealth heterogeneity)***
The reform at time at t_R is a No Regret fiscal reform if and only if there exists a constant H such that for all asset level a, U(a)=(a)+H

Without heterogeneity, the need for a change in the fiscal policies comes from a change in the *preferences regarding government expenditures*. With wealth heterogeneity, such a change may also come from a change in the *preferences regarding redistribution*. However, this theorem establishes that a No Regret reform is not able to achieve more (nor less) redistribution across agents who differ only by their wealth levels.

Equal Sacrifice principle. According the Equal Sacrifice principle, households facing a common burden (here an increase in government expenditures) should share it such that the utility loss is equal among them. This principle has a clear normative appeal of equity. This paper provides another rationale: the Equal Sacrifice principle should be respected along the wealth heterogeneity dimension to make sure that households do not regret their past saving efforts. A symmetric Equal Benefit principle should be respected when there is a common benefit to be shared.

5.2 Skill heterogeneity

I now introduce skill heterogeneity i.e. the disutility of labour becomes skill dependent $v(L, \omega)$ where L is the *efficient* labour effort and $\omega \in [\underline{\omega}, \overline{\omega}]$ is the skill. As usual, I assume that $\frac{\partial v(L,\omega)}{\partial \omega} < 0$. A households' *type* is her skill ω and her initial wealth level a_0. I denote $f(\cdot, \cdot)$ the joint density of types. The marginal density of skill is denoted $f_\omega(\cdot)$. To keep the model tractable with two heterogeneity dimensions, I take a mechanism design approach where the government directly selects consumption $C(\omega, a_0)$ and efficient labour effort $L(\omega, a_0)$ for each *type* (ω, a_0). To simplify, I make several assumptions. First, production equals effective labour and there is a storage technology with returns $1/\beta$. Second, government's budget is balanced. Third, consumption levels and labour efforts are time-invariant. Thus, the flow utility for each type is

$$U(\omega, a_0) \equiv u(C(\omega, a_0)) - v(L(\omega, a_0), \omega)$$

No Regret fiscal reform. A No Regret fiscal reform is implemented at time t_R. The new plan is denoted $\tilde{C}(\cdot, \cdot), \tilde{L}(\cdot, \cdot)$ and the new utility flows are denoted $\tilde{U}(\cdot, \cdot)$. The government does not observe skills ω. Truthful reporting of skills by households implies the following constraint on $\tilde{L}(\cdot, \cdot)$ and $\tilde{U}(\cdot, \cdot)$.

$$\frac{\partial \tilde{U}}{\partial \omega}(\omega, a_0) = -\frac{\partial v(\tilde{L}(\omega, a_0), \omega)}{\partial \omega}$$

The previous subsection established that, when a No Regret fiscal reform is implemented, all households with the same skill must have the same utility loss or gain, whatever their wealth levels. This means that there is a function $\tilde{H}(\cdot)$ such that, for all skill ω and for all initial wealth level a_0,

$$\tilde{U}(\omega, a_0) = U(\omega, a_0) + \tilde{H}(\omega)$$

Thus, it is without loss of generality to use Pareto welfare weights which are *skill dependent* rather than *type dependent*. They are denoted $\Omega(\cdot)$. The problem faced by the government at time t_R is

$$\max_{\substack{\tilde{C}(\cdot,\cdot),\tilde{L}(\cdot,\cdot),\\ \tilde{U}(\cdot,\cdot),\tilde{H}(\cdot),G}} \int \Omega(\omega)\tilde{H}(\omega)f_\omega(\omega)d\omega \; + \; \theta w(G)$$

$$\textit{subject to} \quad \int \tilde{C}(\omega, a_0)f(\omega, a_0)d\omega da_0 + G \leq \int \tilde{L}(\omega, a_0)f(\omega, a_0)d\omega da_0 + (1/\beta - 1)\int a_0 f(\omega, a_0)d\omega da_0$$

$$\textit{to} \quad \tilde{U}(\omega, a_0) = U(\omega, a_0) + \tilde{H}(\omega)$$

$$\textit{to} \quad \frac{\partial \tilde{U}}{\partial \omega}(\omega, a_0) = -\frac{\partial v(\tilde{L}(\omega, a_0), \omega)}{\partial \omega}$$

$$\textit{and to} \quad \tilde{U}(\omega, a_0) = u(\tilde{C}(\omega, a_0)) - v(\tilde{L}(\omega, a_0), \omega)$$

5.3 Tax implementation

General case. To implement the new allocation the government uses *skill dependent* wealth tax schedules at the time of the reform t_R and *wealth dependent* labour tax schedules at all times $t \geq t_R$. For each skill ω, the wealth tax schedule is denoted $\tau_\omega^K(a_0)$. The return on wealth is equal to the discount factor so the Ex Post Euler Equation is

$$u'(C(\omega, a_0)) = [1 - (\tau_\omega^K)'(a_0)]u'(\tilde{C}(\omega, a_0))$$

I assume that households without wealth are not taxed and so one can integrate the previous expression and get for any wealth a_0

$$\tau_\omega^K(a_0) = a_0 - \int_0^{a_0} \frac{u'(C(\omega, s))}{u'(\tilde{C}(\omega, s))} ds$$

Using the households' budget constraints, I get that, for each type (ω, a_0), the labour tax is

$$\tau_{a_0}^L(\omega) = \tilde{L}(\omega, a_0) - \tilde{C}(\omega, a_0) + (1/\beta - 1)[a_0 - \tau_\omega^K(a_0)]$$

Using $\omega_{a_0}^{-1}(\cdot)$, the inverse function of the increasing functions $\omega \mapsto L(\omega, a_0)$, one can rewrite the *skill dependent* tax schedules on wealth as a function of the labour effort *before the reform* denoted L

$$\tau_L^K : a_0 \mapsto \begin{cases} a_0 - \int_0^{a_0} \frac{u'(C(\omega_{a_0}^{-1}(L), s))}{u'(\tilde{C}(\omega_{a_0}^{-1}(L), s))} ds & if \;\; L \in L([\underline{\omega}, \overline{\omega}], a_0) \\ a_0 & otherwise \end{cases}$$

I denote a_{t_R} the after tax wealth. Using $\tilde{\omega}_{a_0}^{-1}(\cdot)$, the inverse function of the increasing functions $\omega \mapsto \tilde{L}(\omega, a_0)$, one can rewrite the *wealth dependent* labour tax schedule for each after tax wealth a_{t_R}.

$$\tau^L_{a_{t_R}} : \tilde{L} \mapsto \begin{cases} \tilde{L} - \tilde{C}(\tilde{\omega}_{a_0}^{-1}(\tilde{L}), a_0) + (1/\beta - 1)a_{t_R} & if \;\; \tilde{L} \in \tilde{L}([\underline{\omega}, \overline{\omega}], a_0) \\ \tilde{L} & otherwise \end{cases}$$

Note that the *wealth dependent* labour tax schedule is a function of the net of tax wealth a_{t_R} *at the time of the reform* t_R. Households cannot change their labour tax schedules by changing their wealth *after the reform*.

Wealth independent labour tax schedule. In the previous paragraph, I described the taxes on labour and capital used to implement a given allocation. It is also possible to choose first *wealth dependent* labour tax schedules and to then find *skill dependent* wealth tax schedules to make sure the reform respects the No Regret rule. As an interesting special case, it is possible to use the *wealth independent* labour tax schedule. However,

5.4 More heterogeneity dimensions

It should be noted that there are many more dimensions of heterogeneity with economic relevance (e.g. education, discount preferences, altruism, age, elasticity of intertemporal substitution, Frish elasticity) and that a government with limited information and simple fiscal instruments cannot announce a reform which is No Regret for all households. However, the No Regret fiscal rule can be adapted to more realistic environments. For instance, I propose to use a characterization of No Regret fiscal reform introduced in lemma 3. To do so, one can divide society in homogeneous groups of agents with similar wealth and skill levels and impose that a reform is such that there is no jump in average consumption or labour within each group when the reform is announced.

6 Simulations

6.1 Specifications

In this section, I describe numerical simulations of the optimal policy under the No Regret rule. Let us first describe the specifications I use.

Technology and stochastic preferences. I use a Cobb-Douglas production function with capital share α and with depreciation of capital at rate δ.

$$F(K, L) = AK^{\alpha}L^{1-\alpha} + (1 - \delta)K$$

I use iso-elastic preferences regarding consumption, labour and governement spending

$$u(C) - v(L) + \theta w(G) = \frac{C^{1-\sigma} - 1}{1-\sigma} - \kappa \frac{L^{1+\chi} - 1}{1+\chi} + \theta \frac{G^{1-\xi} - 1}{1-\xi}$$

The shock process θ is modeled as a two state Markov chain with a constant transition matrix. The two states are denoted H and L (High and Low). The transition matrix is

TRANSITION MATRIX

$\mathbb{P}[\ \theta_{t+1} = H \mid \theta_t = H\ ] \quad = p_{HH}$	$\mathbb{P}[\ \theta_{t+1} = L \mid \theta_t = H\ ] \quad = 1 - p_{HH}$
$\mathbb{P}[\ \theta_{t+1} = L \mid \theta_t = L\ ] \quad = p_{LL}$	$\mathbb{P}[\ \theta_{t+1} = H \mid \theta_t = L\ ] \quad = 1 - p_{LL}$

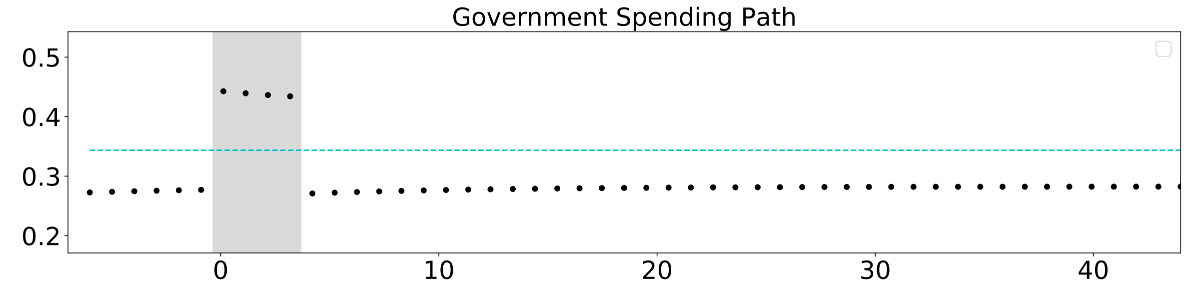
Parameters. I use the following parameters:

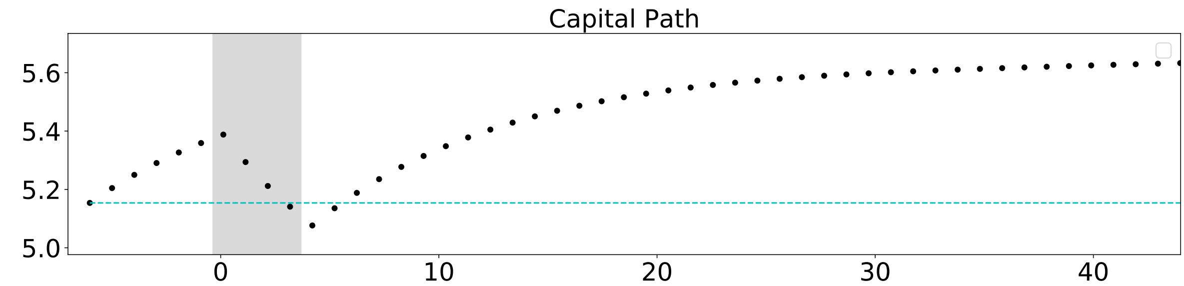
CALIBRATED PARAMETERS		
Discount factor β	0.96	-
Intensive Frisch elasticity $1/\chi$	0.5	Chetty, Guren, Manoli, and Weber (2011)
Private relative risk aversion σ	2	Calvet, Campbell, Gomes, and Sodini (2021)
Public relative risk aversion ξ	2	Identical to σ
Capital share α	0.33	-
p_{HH}	$3/4$	-
p_{LL}	$5/6$	-
δ	6.83 %	K/Y=3
κ	0.72	Normalization Labour 1
$\mathbb{E}[\theta_t]$	0.10	Gov share of GDP = 20 %

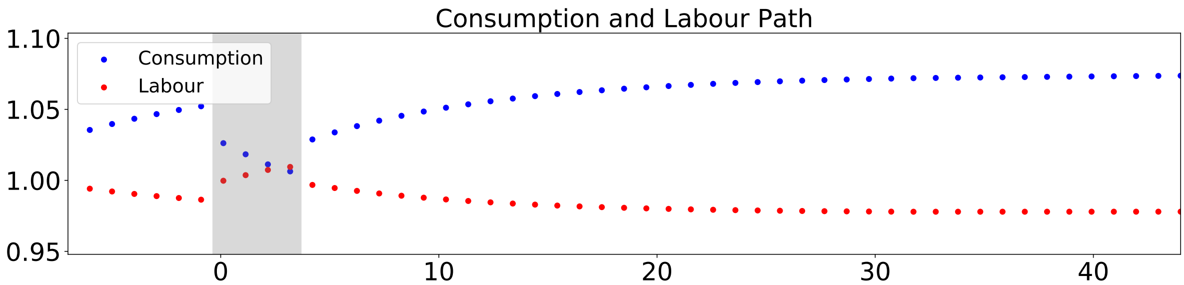
6.2 Allocation

Let me now describe the policy implemented by a government under the No Regret rule. I show the realized allocation and policy for the following realization of the shock process θ. The economy starts at the steady state without shocks. Then, for 6 periods, shocks are low i.e. $\theta_t = L$. Then the shocks are high for 4 periods (grey zone). Finally, shocks are low until the end of time.

The optimal allocation is the following:



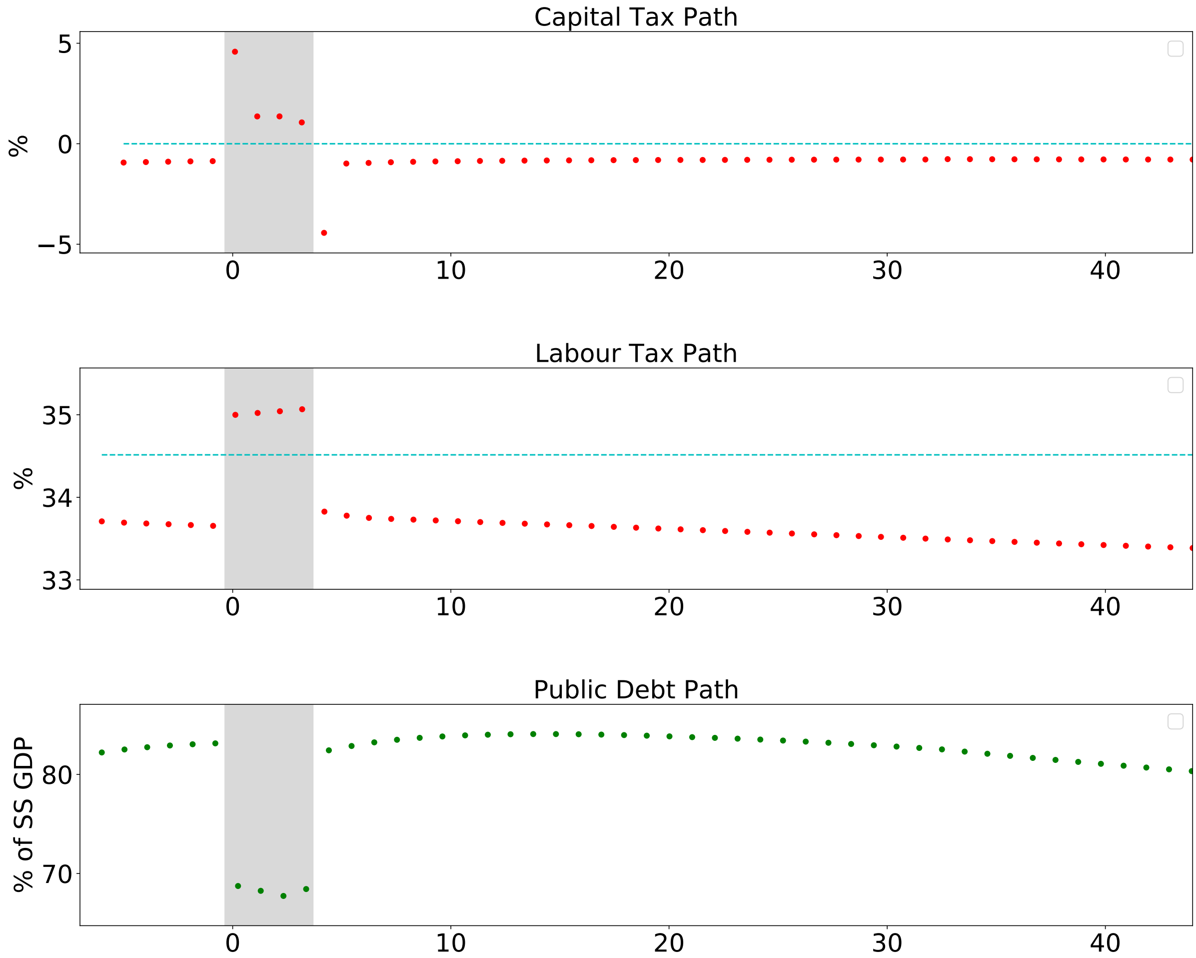




Let me describe how the economy copes with these shocks. First, when the shocks are high, government spending is more useful and the government raises its spending. Second, since preference regarding consumption are constant whereas those regarding government spending vary, the global spending need of the economy varies with shocks and also does the value of production goods. Accordingly it is beneficial that the capital stock decreases when shocks are high and increases when there are low. In other words, capital is used as a buffer to cope with spending shocks. However, capital depreciates and its returns are decreasing. Thus, maintaining a high level of capital is costly. For this reason, there is an upper limit for the capital stock towards which the economy converges when shocks keep being low. Symmetrically, a low level of capital implies high returns on capital. For this reason there is a lower limit to the capital stock towards which the economy converges when shocks keep being high (not shown in these simulations). Third, since the value of production goods varies with shocks, the labour effort is higher when shocks are high. Forth, since the preference for government spending *relative* to consumption is higher when shocks are high, the consumption level is low. Ceteris paribus, consumption and government spending are higher and labour is lower when the economy is richer (i.e. capital stock is higher). The next subsection describes the tax implementation of this allocation.

6.3 Tax implementation

How does the government under the No Regret rule implement the allocation have just described ? How to make households choose the desired consumption, labour and capital levels ? How to fund the government expenditures ? How to make sure the No Regret rule is respected ? The fiscal policy which implements the previous allocation is the following:



On impact. When the shock goes from low to high, the government raises significantly its spending level. To make households consume less, invest less in capital and work more the government relies on wealth effect. In other words, it wants to make households poorer. To do so it uses two tools, the capital tax and the labour tax. The capital tax decreases households current wealth and the labour tax decreases net labour incomes. When the first high shock realizes, the capital tax[13] is set at a high rate (5%). This tax decreases households current wealth and the associated tax revenues are used to pay back part of the public debt which drops. The labour tax is also raised when the first high shock realizes. This labour tax hike is persistent i.e. even when the high shocks are over, the labour tax is higher than what it would have been without high shocks and with only low shocks instead. This persistence implies that the current *and* future net labour incomes go down which magnifies the wealth effect. This wealth effect leads households to reduce their consumption and their investment in capital, and to increase their labour efforts. Symmetrically, when the shock goes from high to low, the government relies on a subsidy on capital and on a persistent labour tax drop to make households richer. This leads them to consume more, to invest more in capital and to work less.

High shock zone. When shocks stay high for a few years, the capital tax is kept positive and the labour tax slowly increases. These make households wealth slowly decrease and so they choose to slowly reduce consumption and investment in capital while slowly increasing work effort.

Low shock zone. When shocks keep being low, capital keeps being subsidized. This subsidy *ex ante* offsets the distortions of the high capital tax in the case a high shock were to realize. This appears even more clearly in the long run, when consumption converges. The capital stock becomes high and the gross returns on capital are lower than the discount rate. A subsidy on capital is needed to make sure that the Ex Post Euler Equation holds and that households don't regret their past decisions. After some years, capital stock is too high and high gross returns are too low. The subsidy becomes insufficient to make households richer so they don't increase their consumption nor their capital stock. Furthermore, since capital increases, the gross wage also increases. This raises the proceeds of the labour tax which allow the government to slowly run down its debt.

6.4 Tax distortions

Since distortion costs are convex in the labour tax rate, it is optimal to keep the labour tax as smooth as possible. Ideally, the labour tax should be constant across periods and across realizations of shocks.[14] The labour tax increases when shocks are high. This increase is larger when the previous shocks are low. Indeed, since shocks are persistent, the following shocks are likely to be high as well which means that the expected

[13] The tax on bonds follows a similar path

[14] The labour tax under the optimal contingent policy is constant across periods and across realizations of shocks

spending paths shifts up dramatically and so does the government funding needs. When previous shock are high, the expected spending paths shifts up but less dramatically. Symmetrically, the labour tax decreases when a low shock realizes. However, the capital tax is such that the wedge to savings is close to zero so saving decisions are hardly distorted. This means that the associated distortion cost is also close to zero. It is worth mentioning that it is not the capital tax distortions to savings that makes households decrease or increase the capital stock as the government desires, it is the wealth effect of the capital tax and of the labour tax.

6.5 Welfare

Three mechanisms. Under the No Regret rule, the government uses three mechanisms to cope with shocks: it varies the labour tax, it runs budget surplus or deficits, and it uses the capital stock as a buffer. Are these mechanisms equivalently useful to adapt to spending shocks ? I froze one or two mechanisms and computed the associated welfare losses in consumption equivalent.

WELFARE DIFFERENCES (in consumption equivalent)	
Optimal No Regret policy	-
... With Constant Capital i.e. $K_t = K_{SS}$	-0.05%
... With Balanced Budget i.e. $B_t = 0$	-0.04%
... And Constant Capital i.e. $K_t = K_{SS}$	-0.49%
... With Constant Labour Tax i.e. $\tau_t^L = \tau_{SS}^L$	-0.08%
... And Constant Capital i.e. $K_t = K_{SS}$	-0.71%
Optimal Constant Policy	-1.20%

The two main take-aways are the following. First, with this specification, the three mechanisms appear similarly useful to adapt to shocks. Second, freezing one out of the three mechanisms is not very costly in terms of welfare.

Optimal contingent policy. Another important result is that the difference in welfare levels achieved by the optimal contingent policy and the optimal No Regret policy is very small, less than 0.01% in consumption equivalent.

7 Conclusion

The main contribution of this paper is to introduce No Regret fiscal reforms ans the No Regret rule which offer a convincing alternative to contingent policies defined once and for all. I have shown that the small utility loss compared to the optimal contingent plan is due to the fact that the government cannot get full insurance through contingent bonds. I have studied optimal contingent policies robust to Near-Rational Expectations. I also established that the welfare difference between the optimal robust contingent policy and the optimal No Regret policy disappears when the robustness requirement is high enough. Finally, I introduced wealth and skill heterogeneity and showed that, under the No Regret rule, redistribution is possible across skills but not across wealth levels.

Several avenues for future research are worth mentioning. First, in this paper, the availability of No Regret reforms relies on the fact that the government knows perfectly households' preferences and firms' technology. Looking for "Regret" sufficient statistics associated to each policy tools that summarize what the government needs to know to guarantee the absence of regret could be promising. Second, in this paper it is assumed that governments are infinitely sophisticated in the sense that they know all future shocks and their joint probability distributions. It could be interesting to relax this assumption by introducing unforeseen contingencies. Contingent policies won't be able to adapt to these shocks whereas a government under the No Regret rule could react. This would make of the No Regret rule an even more convincing alternative to contingent policies.

References

Abo-Zaid, S. (2014). Optimal capital-income taxation in a model with credit frictions. *The BE Journal of Macroeconomics*, *14*(1), 147–172.

Aiyagari, S. R. (1995). Optimal capital income taxation with incomplete markets, borrowing constraints, and constant discounting. *Journal of political Economy*, *103*(6), 1158–1175.

Aiyagari, S. R., Marcet, A., Sargent, T. J., & Seppälä, J. (2002). Optimal taxation without state-contingent debt. *Journal of Political Economy*, *110*(6), 1220–1254.

Angeletos, G.-M. (2002). Fiscal policy with noncontingent debt and the optimal maturity structure. *The Quarterly Journal of Economics*, *117*(3), 1105–1131.

Athey, S., Atkeson, A., & Kehoe, P. J. (2005, September). The Optimal Degree of Discretion in Monetary Policy. *Econometrica*, *73*(5), 1431-1475. Retrieved from https://ideas.repec.org/a/ecm/emetrp/v73y2005i5p1431-1475.html

Atkinson, A. B., & Stiglitz, J. E. (1976). The design of tax structure: direct versus indirect taxation. *Journal of public Economics*, *6*(1-2), 55–75.

Boar, C., & Knowles, M. (2018). *Entrepreneurship, agency frictions and redistributive capital taxation* (Tech. Rep.). Working paper, New York University.

Buera, F., & Nicolini, J. P. (2004). Optimal maturity of government debt without state contingent bonds. *Journal of Monetary Economics*, *51*(3), 531–554.

Calvet, L. E., Campbell, J. Y., Gomes, F., & Sodini, P. (2021). The cross-section of household preferences.

Chamley, C. (1986). Optimal taxation of capital income in general equilibrium with infinite lives. *Econometrica: Journal of the Econometric Society*, 607–622.

Chari, V. V., Christiano, L. J., & Kehoe, P. J. (1994). Optimal fiscal policy in a business cycle model. *Journal of political Economy*, *102*(4), 617–652.

Chetty, R., Guren, A., Manoli, D., & Weber, A. (2011). Are micro and macro labor supply elasticities consistent? a review of evidence on the intensive and extensive margins. *American Economic Review*, *101*(3), 471–75.

Debortoli, D., & Nunes, R. (2010). Fiscal policy under loose commitment. *Journal of Economic Theory*, *145*(3), 1005–1032.

Debortoli, D., & Nunes, R. (2013). Lack of commitment and the level of debt. *Journal of the European Economic Association*, *11*(5), 1053–1078.

Debortoli, D., Nunes, R., & Yared, P. (2017). Optimal time-consistent government debt maturity. *The Quarterly Journal of Economics*, *132*(1), 55–102.

Dovis, A. (2019). Efficient sovereign default. *The Review of Economic Studies*, *86*(1), 282–312.

Erosa, A., & Gervais, M. (2002). Optimal taxation in life-cycle economies. *Journal of Economic Theory*, *105*(2), 338–369.

Farhi, E. (2010). Capital taxation and ownership when markets are incomplete. *Journal of Political Economy*, *118*(5), 908–948.

Farhi, E., Sleet, C., Werning, & Yeltekin, S. (2012). Non-linear capital taxation without commitment. *Review of Economic Studies*, *79*(4), 1469–1493.

Farhi, E., & Werning. (2007). Inequality and social discounting. *Journal of political economy*, *115*(3), 365–402.

Farhi, E., & Werning. (2010). Progressive estate taxation. *The Quarterly Journal of Economics*, *125*(2), 635–673.

Gerritsen, A., Jacobs, B., Rusu, A. V., & Spiritus, K. (2020). Optimal taxation of capital income with heterogeneous rates of return.

Gervais, M., & Mennuni, A. (2015). Optimal fiscal policy in the neoclassical growth model revisited. *European Economic Review*, *73*, 1–17.

Guo, J.-T., & Lansing, K. J. (1999). Optimal taxation of capital income with imperfectly competitive product markets. *Journal of Economic Dynamics and Control*, *23*(7), 967–995.

Halac, M., & Yared, P. (2014). Fiscal rules and discretion under persistent shocks. *Econometrica*, *82*(5), 1557–1614.

Judd, K. L. (1985). Redistributive taxation in a simple perfect foresight model. *Journal of public Economics*, *28*(1), 59–83.

Judd, K. L. (2002). Capital-income taxation with imperfect competition. *American Economic Review*, *92*(2), 417–421.

Karantounias, A. G. (2013). Managing pessimistic expectations and fiscal policy. *Theoretical Economics*, *8*(1), 193–231.

Klein, P., & Ríos-Rull, J.-V. (2003). Time-consistent optimal fiscal policy. *International Economic Review*, *44*(4), 1217–1245.

Kocherlakota, N. R. (2005). Zero expected wealth taxes: A mirrlees approach to dynamic optimal taxation. *Econometrica*, *73*(5), 1587–1621.

Krusell, P., Martin, F. M., Ríos-Rull, J.-V., et al. (2006). Time consistent debt. In *2006 meeting papers*.

Kydland, F. E., & Prescott, E. C. (1977). Rules rather than discretion: The inconsistency of optimal plans. *Journal of political economy*, *85*(3), 473–491.

Loomes, G., & Sugden, R. (1982). Regret theory: An alternative theory of rational choice under uncertainty. *The economic journal*, *92*(368), 805–824.

Lucas Jr, R. E., & Stokey, N. L. (1983). Optimal fiscal and monetary policy in an economy without capital. *Journal of monetary Economics*, *12*(1), 55–93.

Piketty, T., & Saez, E. (2013). A theory of optimal inheritance taxation. *Econometrica*, *81*(5), 1851–1886.

Reis, C. (2011). Entrepreneurial labor and capital taxation. *Macroeconomic Dynamics*, *15*(3), 326.

Saez, E. (2002). The desirability of commodity taxation under non-linear income taxation and heterogeneous tastes. *Journal of Public Economics*, *83*(2), 217–230.

Saez, E., & Stantcheva, S. (2018). A simpler theory of optimal capital taxation. *Journal of Public Economics*, *162*, 120–142.

Savage, L. J. (1951). The theory of statistical decision. *Journal of the American Statistical association*, *46*(253), 55–67.

Sleet, C. (2004). Optimal taxation with private government information. *The Review of Economic Studies*, *71*(4), 1217–1239.

Smith, M., Yagan, D., Zidar, O., & Zwick, E. (2019). Capitalists in the twenty-first century. *The Quarterly Journal of Economics*, *134*(4), 1675–1745.

Straub, L., & Werning, I. (2020). Positive long-run capital taxation: Chamley-judd revisited. *American Economic Review*, *110*(1), 86–119.

Svec, J. (2012). Optimal fiscal policy with robust control. *Journal of Economic Dynamics and Control*, *36*(3), 349–368.

Woodford, M. (2006). An example of robustly optimal monetary policy with near-rational expectations. *Journal of the European Economic Association*, *4*(2-3), 386–395.

Woodford, M. (2010). Robustly optimal monetary policy with near-rational expectations. *American Economic Review*, *100*(1), 274–303.
Zhu, X. (1992). Optimal fiscal policy in a stochastic growth model. *Journal of Economic Theory*, *58*(2), 250–289.

8 Proofs

Necessity of (OL), (eaEE) and (eaEE')

Back to main text
Assume $\{C^\infty, L^\infty, q^\infty, k^\infty\}$ is optimal, it is budget feasible so (HBC) hold.
We have assumed that the debt constraint was never binding so one can build small deviations to establish (OL), (eaEE), (eaEE') by contradiction.

(OL) : If $v'(L_t) < W_t u'(C_t)$ then there is $\varepsilon > 0$ such that a deviation with $\tilde{L}_t = L_t + \varepsilon$ and $\tilde{C}_t = C_t + W_t\varepsilon$ yields higher welfare to the households and (HBC) still holds, contradicting optimality. The reverse inequality can be ruled out with $\varepsilon < 0$.

(eaEE) : If $u'(C_t) < \mathbb{E}_t[\, \beta R_{t+1} u'(C_{t+1})\,]$ then there is $\varepsilon > 0$ such that a deviation with $\tilde{k}_{t+1} = k_{t+1} + \varepsilon$, $\tilde{C}_t = C_t - \varepsilon$ and $\tilde{C}_{t+1} = C_{t+1} + R_{t+1}\varepsilon$ yields higher welfare to the households and (HBC) still holds, contradicting optimality. The reverse inequality can be ruled out with $\varepsilon < 0$.

(eaEE') : $P_t < \frac{\mathbb{E}_t[\, \beta B_{t+1} u'(C_{t+1})\,]}{u'(C_t)}$ then there is $\varepsilon > 0$ such that a deviation with $\tilde{q}_t = q_t + \varepsilon$, $\tilde{C}_t = C_t - \varepsilon P_t$ and $\tilde{C}_{t+1} = C_{t+1} + B_{t+1}\varepsilon$ yields higher welfare to the households and (HBC) still holds, contradicting optimality. The reverse inequality can be ruled out with $\varepsilon < 0$.

■

Lemma ??: Partial Characterization of No Regret allocations

Back to main text
⇒Take prices and bonds $\{W^\infty, R^\infty, P^\infty, B^\infty\}$ such that (HBC), (OL), (eaEE), (eaEE') hold at all time t.
Define for all time t, $D_t \equiv (R_t k_t + q_{t-1} B_t)$
So at all time t, $C_t + k_{t+1} + q_t P_t = R_t k_t + q_{t-1} B_t + W_t L_t + T_t$
Multiply by $u'(C_t)$ and use (OL), (eaEE), (eaEE') and $T_t \geq 0$ to get:

$$u'(C_t)C_t + \mathbb{E}_t[\, \beta R_{t+1} u'(C_{t+1})\,] k_{t+1} + \mathbb{E}_t[\, \beta B_{t+1} u'(C_{t+1})\,] q_t \geq u'(C_t) R_t k_t + u'(C_t) q_{t-1} B_t + v'(L_t) L_t + u'(C_t) T_t$$

So

$$u'(C_t)D_t \le u'(C_t)C_t - v'(L_t)L_t + \mathbb{E}_t[\,\beta u'(C_{t+1})D_{t+1}\,]$$

$\Leftarrow$ Define $W_t \equiv \frac{v'(L_t)}{u'(C_t)}$ such that (OL) holds
Define $R_{t+1} \equiv \frac{u'(C_t)}{\beta u'(C_{t+1})}$ such that (eaEE) holds
Define $B_t \equiv \frac{D_t - R_t k_t}{q_{t-1}}$ and $P_t = \frac{\mathbb{E}_t[\,\beta B_{t+1} u'(C_{t+1})\,]}{u'(C_t)}$ such that (eaEE') holds.
Assume there exists D^∞ such that at all time t,
$u'(C_t)D_t \le u'(C_t)C_t - v'(L_t)L_t + \mathbb{E}_0[\,\beta u'(C_{t+1})D_{t+1}\,]$
By definition of B_t

$$u'(C_t)[R_t k_t + q_{t-1}B_t] \le u'(C_t)C_t - v'(L_t)L_t + \mathbb{E}_t[\,\beta u'(C_{t+1})[R_{t+1}k_{t+1} + q_t B_{t+1}]\,]$$

Divide by $u'(C_t) > 0$ and use (OL), (eaEE) and (eaEE'):

$$R_t k_t + q_{t-1}B_t \le C_t - W_t L_t + k_{t+1} + q_t P_t$$

There exist a positive transfer T_t such that (HBC) holds.

■

Sufficiency of (OL), (eaEE), (eaEE') and (TR)

Back to main text
Assume $\{C^\infty, L^\infty, q^\infty, k^\infty\}$ is such that (OL),(eaEE), (eaEE') and (TR) hold
Denote $\{\tilde{C}^\infty, \tilde{L}^\infty, \tilde{q}^\infty, \tilde{k}^\infty\}$ another behaviour and denote:

$$\Delta_T \equiv \mathbb{E}_0[\,\sum_{t=0}^{T}\beta^t\,[\,u(\tilde{C}_t) - v(\tilde{L}_t)\,]\,\,] \;-\; \mathbb{E}_0[\,\sum_{t=0}^{T}\beta^t\,[\,u(C_t) - v(L_t)\,]\,\,]$$

$$\le \mathbb{E}_0[\,\sum_{t=0}^{T}\beta^t\,[\,u'(C_t)(\tilde{C}_t - C_t) - v'(L_t)(\tilde{L}_t - L_t)\,]\,\,]$$

$$\le \mathbb{E}_0[\,\sum_{t=0}^{T}\beta^t\,[\,u'(C_t)(R_t(\tilde{k}_t - k_t) + B_t(\tilde{q}_{t-1} - q_{t-1}) - (\tilde{k}_{t+1} - k_{t+1}) - (\tilde{q}_t - q_t)P_t + W_t(\tilde{L}_t - L_t)) - v'(L_t)(\tilde{L}_t - L_t)\,]\,]$$

(OL) holds for all time t so, $u'(C_t)W_t(\tilde{L}_t - L_t) - v'(L_t)(\tilde{L}_t - L_t) = 0$
(eaEE) and (eaEE') holds so one gets

$$\Delta_T \le -\mathbb{E}_0[\,\beta^T u'(C_T)\{(\tilde{k}_{T+1} - k_{T+1}) + (\tilde{q}_T - q_T)P_T\}\,]$$

$$\Delta_T \le \mathbb{E}_0[\,\beta^T u'(C_T)\{k_{T+1} + q_T P_T + M\} - \beta^T u'(C_T)\{(\tilde{k}_{T+1} + \tilde{q}_T P_T + M\}\,]$$

Since (TR) holds, $\mathbb{E}_0[\,\beta^T u'(C_T)\{k_{T+1} + q_T P_T + M\}\,] \underset{T\to\infty}{\longrightarrow} 0$
The other term is negative so

$$\underset{T\to\infty}{\underline{\lim}} \;\; \mathbb{E}_0[\,\sum_{t=0}^{T} \beta^t\,[\,u(\tilde{C}_t) - v(\tilde{L}_t)\,]\,] \;-\; \mathbb{E}_0[\,\sum_{t=0}^{T} \beta^t\,[\,u(C_t) - v(L_t)\,]\,] \;\le\; \underset{T\to\infty}{\overline{\lim}}\;\Delta_T \le 0$$

So indeed $\{C^\infty, L^\infty, q^\infty, k^\infty\}$ is optimal.

■

Lemma 3: No Regret reform characterization

Back to main text

1) $\Rightarrow$ 2) By definition of a No Regret reform, households don't regret their past behaviours so whatever the time $t^i_R \;\le t_R$, they are informed about the reform, they don't change their behaviours.
2) $\Rightarrow$ 3) holds because 2) is more general
3) $\Rightarrow$ 1) Let us denote $(C_t)_{t\in\mathbb{N}}$ and $(L_t)_{t\in\mathbb{N}}$ the initial planned actions and $(C'_t)_{t\in\mathbb{N}}$ and $(L'_t)_{t\in\mathbb{N}}$ the (past or planned) actions once informed about the reform at $t_R - 1$ or t_R.
When informed at t_R, the household cannot change its past actions, so for all $t \le t_R - 1$, $C_t = C'_t$ and $L_t = L'_t$.
$(C_t)_{t\in\mathbb{N}}$ was optimally chosen at time 0 by the household so (dEE) holds for $(C'_t)_{t\in\mathbb{N}}$ at all periods between 0 and $t_R - 2$.
Since $(C'_t)_{t\in\mathbb{N}}$ is optimally chosen at $t_R - 1$, (dEE) also hold at $t_R - 1$ and at all later periods as well. Furthermore the transversality constraint (TR) also holds.
Since $(L'_t)_{t\in\mathbb{N}}$ is optimally chosen at all periods, (OL) always holds.
Since $(C'_t)_{t\in\mathbb{N}}$ and $(L'_t)_{t\in\mathbb{N}}$ are such that (dEE) (OL) and (TR) hold at all time, these actions must be optimal from time 0 i.e. household couldn't have taken better actions so the reform is No Regret.

■

Lemma 4: No Regret allocations insensitive to households beliefs

Back to main text

Thanks to lemma **??**, the ex post Euler Equation holds at any time t, i.e. $u'(C_t) = \beta R_{t+1} u'(C_{t+1})$.
This means that for any beliefs p^B,

$$u'(C_t) = \mathbb{E}^B_t[\,\beta R_{t+1} u'(C_{t+1})\,] := \int_{\omega\in\Omega} \beta R_{t+1}(\omega) u'(C_{t+1}(\omega)) dp^B(\omega)$$

This means that households' behaviours is insensitive to their beliefs $\{p^B_i\}_i$. So, the equilibrium allocation associated to the No Regret policy plan is insensitive to households' beliefs.

■

Lemma ?? : Deviations and Government utility change

Back to main text

At t=0, the prices are announced and the agents deviate from the optimal No Regret plan: $\{(dC_n)_{1\leq n\leq N}, (dL_n)_{1\leq n\leq N}\}$.

By assumption, the net of tax wages are unchanged compared to the No Regret plan so

$$\frac{v'(L_n)}{u'(C_n)} = \frac{v'(L_n + dL_n)}{u'(C_n + dC_n)} (= W_n)$$

Which gives the following first order approximation:

$$\sigma_n \frac{dC_n}{C_n} = -\chi_n \frac{dL_n}{L_n}$$

I define $x_n := \sigma_n \frac{dC_n}{C_n}$
Similarly at $t = 0$ $x_0 := \sigma_0 \frac{dC_0}{C_0}$

Since $\{C_0, (C_n)\}$ is part of a No Regret plan the Euler equation between $t = 0$ and $t = 1$ holds for each state n i.e. for all n

$$u'(C_0) = R_n u'(C_n)$$

In the deviations the (expected) Euler Equation holds so using agents beliefs p^B

$$u'(C_0 + dC_0) = \sum_{n=1}^{N} p_n^B [\ R_n(1 + dr_n)u'(C_n + dC_n)\ ]$$

Which gives the following first order approximation:

$$u''(C_0)dC_0 = \sum_{n=1}^{N} p_n^B [\ u'(C_0)dr_n + u''(C_n)\frac{u'(C_0)}{u'(C_n)} dC_n\ ]$$

$$-x_0 = \sum_{n=1}^{N} p_n^B [\ dr_n - x_n\ ]$$

By assumption, the past promises are kept so it pins down the net of tax returns at $t = 0$ and we get: $u'(C_{-1}) = R_0 u'(C_0) = R_0(1+dr_0)u'(C_0+dC_0) So$u''(C$_0$)$dC_0+u'(C_0)dr_0 = 0$ and

$$dr_0 = -\sigma_0 \frac{dC_0}{C_0} = -x_0$$

The last equation one needs comes from the N budget constraints of the households:

$$(A_0 - C_0 + W_0 L_0)R_n - C_n + W_n L_n = 0$$

$$(A_0 - C_0 - dC_0 + W_0(L_0 + dL_0))R_n(1 + dr_n) - C_n - dC_n + W_n(L_n + dL_n)$$

Combining these two equations one gets

$$(-dC_0 + W_0 dL_0)R_n + A_1 R_n dr_n - dC_n + W_n dL_n = 0$$

Multiplying this equation by $u'(C_n)$ and using the Euler equations and $W_n = \frac{v'(L_n)}{u'(C_n)}$

$$-u'(C_0)dC_0 + v'(L_0)dL_0 + A_1 u'(C_0)dr_n - u'(C_n)dC_n + v'(L_n)dL_n = 0$$

Then using the definitions of $E_0,\ E_n,\ x_0,\ x_n,\ I_0,\ I_n$ we get :

$$E_0 x_0 + E_n x_n = I_1 dr_n$$

So, for the deviations (dr_n), $x_0, (x_n)$ are given by this system of equations:

$$\begin{cases} -x_0 + \sum_{n=1}^N p_n^B x_n = \sum_{n=1}^N p_n^B dr_n \\ E_0 x_0 + E_n x_n = I_1 dr_n \end{cases} . \tag{9}$$

Solving first for x_0

$$x_0 = -\frac{\sum_{n=1}^N p_n^B dr_n - I_1 \sum_{n=1}^N p_n^B \frac{dr_n}{E_n}}{1 + \sum_{n=1}^N p_n^B \frac{E_0}{E_n}}$$

$$x_n = -\frac{E_0}{E_n} x_0 + I_1 \frac{dr_n}{E_n}$$

Let us now consider the welfare effect of this reform.

$$\begin{aligned} dU_G = (u'(C_0) - W'(G_0))dC_0 + (F_L^0 W'(G_0) - v'(L_0))dL_0 \\ + \sum_{n=1}^N p_n[(u'(C_n) - W'(G_n))dC_n + (F_L^n W'(G_n) - v'(L_n))dL_n] \end{aligned} \tag{10}$$

Using the fact that $\{(C_n)_{1\leq n\leq N}, (L_n)_{1\leq n\leq N}, (G_n)_{1\leq n\leq N}\}$ is an optimal No Regret plan, we get that

$$\lambda_0 = \frac{\theta_0 w'(G_0) - u'(C_0)}{-u''(C_0)A_0 + (1-\sigma_0)u'(C_0)} = \frac{\theta_0 w'(G_0)F_L(K_1, L_0) - v'(L_0)}{(1+\chi_0)v'(L_0)}$$

$$=$$

$$\sum_{n=1}^N p_n \lambda_n = \sum_{n=1}^N p_n[\frac{\theta_n w'(G_n) - u'(C_n)}{(1-\sigma_n)u'(C_n)}] = \sum_{n=1}^N p_n[\frac{\theta_n w'(G_n)F_L(K_1, L_n) - v'(L_n)}{(1+\chi_n)v'(L_n)}]$$

To get

$$\begin{aligned} dU_G = -\sum_{n=1}^N p_n[\ \ -\lambda_n u''(C_0)A_0 dC_0 + \lambda_n \frac{C_0(1-\sigma_0)u'(C_0)}{\sigma_0} x_0 + \lambda_n \frac{L_0(1+\chi)v'(L_0)}{\chi} x_0 \\ + \lambda_n \frac{C_n(1-\sigma_n)u'(C_n)}{\sigma_n} x_n + \lambda_n \frac{L_n(1+\chi_n)v'(L)}{\chi_n} x_n \ \ ] \end{aligned} \tag{11}$$

$$dU_G = -\sum_{n=1}^{N} p_n\lambda_n[\quad u'(C_0)A_0x_0 + E_0x_0 - I_0x_0 + E_nx_n - I_1x_n]$$

$$dU_G = -\sum_{n=1}^{N} p_n\lambda_n[\quad (I_1 + I_0)x_0 + I_1dr_n - I_0x_0 - I_1x_n]$$

$$dU_G = \sum_{n=1}^{N} p_n\lambda_n[\quad -I_1dr_n - I_1x_0 + I_1(\,-\frac{E_0}{E_n}x_0 + I_1\frac{dr_n}{E_n}\,)\quad]$$

$$dU_G = \sum_{n=1}^{N} p_n\lambda_nI_1[\quad -(1-\frac{I_1}{E_n})dr_n - (1+\frac{E_0}{E_n})x_0\quad]$$

$$dU_G = \sum_{n=1}^{N} p_n\lambda_nI_1[\quad -(1-\frac{I_1}{E_n})dr_n\quad] + \sum_{n=1}^{N} p_n\lambda_nI_1[\quad -(1+\frac{E_0}{E_n})\quad](-\frac{\sum_{n=1}^{N} p_n^Bdr_n - I_1\sum_{n=1}^{N} p_n^B\frac{dr_n}{E_n}}{1+\sum_{n=1}^{N} p_n^B\frac{E_0}{E_n}})$$

$\blacksquare$

Lemma ?? : Value function

Back to main text

First, V is decreasing because the set containing p^B grows larger with $delta$. Second, V cannot be strictly negative because non deviating from the optimal No Regret plan, i.e. choosing $dr = 0$, is always available to the government. Third, for all $\delta \geq \delta^\lambda$ setting $p_n^B = \frac{\lambda_np_n}{\sum_{s=1}^{N}\lambda_sp_s}$ is possible. So, for any dr, the following expression is negative.

$$Min_{p^B}\quad -\frac{\mathbb{E}[\quad \lambda\{dr - I_1\frac{dr}{E}\}\quad]}{\mathbb{E}[\quad \lambda\{1+\frac{E_0}{E}\}\quad]} + \frac{\mathbb{E}[\quad \frac{p^B}{p}\{dr - I_1\frac{dr}{E}\}\quad]}{\mathbb{E}[\quad \frac{p^B}{p}\{1+\frac{E_0}{E}\}\quad]}$$

$$subject\ to\quad \sum_{n=1}^{N} p_nln(\frac{p_n^B}{p_n}) \geq -\delta \qquad \sum_{n=1}^{N} p_n^B = 1$$

This establishes that if $\delta \geq \delta^\lambda$ then $V(\delta) = 0$.

Finally, let us show now that if $\delta < \delta^\lambda$ then $V(\delta) > 0$.
Let us denote $\bar{D} := \sum_{n=1}^{N}\frac{\lambda_np_n}{\sum_{s=1}^{N}\lambda_sp_s}(1+\frac{E_0}{E_n})$ and define dr_n^λ[15]:

$$dr_n^\lambda := \frac{\frac{1+\frac{E_0}{E_n}}{\bar{D}} + \bar{D}(\frac{\sum_{s=1}^{N}\lambda_sp_s}{\lambda_n} - 1)}{1 - I_1\frac{1}{E_n}}$$

Let us show that $p_n^B = \frac{\lambda_np_n}{\sum_{s=1}^{N}\lambda_sp_s}$ are solutions to the Min problem.

[15] If $||dr^\lambda|| > 1$ I could multiply it by $\frac{1}{||dr^\lambda||}$.

Note that

$$\sum_{n=1}^{N} p_n^B\{\ (1-I_1\frac{1}{E_n})dr_n^\lambda\ \} = \sum_{n=1}^{N}\frac{\lambda_n p_n}{\sum_{s=1}^{N}\lambda_s p_s}\{\ (1-I_1\frac{1}{E_n})dr_n^\lambda\ \}$$

$$= \sum_{n=1}^{N}\frac{\lambda_n p_n}{\sum_{s=1}^{N}\lambda_s p_s}\{\frac{1+\frac{E_0}{E_n}}{\bar{D}} + \bar{D}(\frac{\sum_{s=1}^{N}\lambda_s p_s}{\lambda_n} - 1)\}$$

$$= \frac{\bar{D}}{\bar{D}} + \bar{D}\sum_{n=1}^{N}\frac{\lambda_n p_n}{\sum_{s=1}^{N}\lambda_s p_s}(\frac{\sum_{s=1}^{N}\lambda_s p_s}{\lambda_n} - 1) = 1 + 0$$

For all n, the Kuhn Tucker conditions, where the shadow values of the entropy constraint and of the probability constraint on p^B are respectively 1 and 1, hold:

$$\frac{dr_n^\lambda(1-I_1\frac{1}{E_n})\bar{D} - (1+\frac{E_0}{E_n})1}{\bar{D}^2} - \frac{p_n}{p_n^B} + 1 = 0$$

$$\Longleftrightarrow$$

$$\frac{1+\frac{E_0}{E_n}}{\bar{D}} + \bar{D}(\frac{\sum_{s=1}^{N}\lambda_s p_s}{\lambda_n} - 1) - \frac{(1+\frac{E_0}{E_n})}{\bar{D}} - \frac{\sum_{s=1}^{N}\lambda_s p_s}{\lambda_n}\bar{D} + \bar{D} = 0$$

So, for a small $\varepsilon > 0$,

$$Min_{p^B}\ -\frac{\mathbb{E}[\ \ \lambda\{dr - I_1\frac{dr}{E}\}\ \ ]}{\mathbb{E}[\ \ \lambda\{1+\frac{E_0}{E}\}\ \ ]} + \frac{\mathbb{E}[\ \ \frac{p^B}{p}\{dr - I_1\frac{dr}{E}\}\ \ ]}{\mathbb{E}[\ \ \frac{p^B}{p}\{1+\frac{E_0}{E}\}\ \ ]}$$

$$subject\ to\ \ \sum_{n=1}^{N} p_n ln(\frac{p_n^B}{p_n}) \geq -(\delta^\lambda - \varepsilon) \quad \sum_{n=1}^{N} p_n^B = 1$$

$$\geq \varepsilon/2 > 0$$

So $V(\delta^\lambda - \varepsilon) \geq \varepsilon/2 > 0$

■